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ABSTRACT

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in summer, 1965. It is intended for use by college students at the Junior and Senior levels. There are nine chapters in this monograph. The failure of classical theory in dealing with elementary particles physics is discussed in chapter 1. Rutherford's experiment on the scattering of a particle and its consequences for theory are presented in chapter 2. Chapter 3 concerns electrons and protons while chapter 4 presents an analysis of electron diffraction. Heisenberg's principle of indeterminacy and its interpretations are the subjects of chapters 5 and 6. The author discusses the dynamical properties of microsystems in chapter 7. The principle of statistical determinism and the microphysical concept of state are presented in chapter 8. Chapter 9 summarizes the basic ideas of quantum mechanics. The monograph concludes with a list of general references for non-classical characteristics of microphysical phenomena. (LC)

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The Conceptual Foundations of Quantum Mechanics

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GENERAL PREFACE

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in the summer of 1965. The general purpose of the conference was to create effective ways of presenting physics to college students who are not preparing to become professional physicists. Such an audience might include prospective secondary school physics teachers, prospective practitioners of other sciences, and those who wish to learn physics as one component of a liberal education.

At the Conference some 40 physicists and 12 filmmakers and designers worked for periods ranging from four to nine weeks. The central task, certainly the one in which most physicists participated, was the writing of monographs.

Although there was no consensus on a single approach, many writers felt that their presentations ought to put more than the customary emphasis on physical insight and synthesis. Moreover, the treatment was to be "multi-level" --- that is, each monograph would consist of several sections arranged in increasing order of sophistication. Such papers, it was hoped, could be readily introduced into existing courses or provide the basis for new kinds of courses.

Monographs were written in four content areas: Forces and Fields, Quantum Mechanics, Thermal and Statistical Physics, and the Structure and Properties of Matter. Topic selections and general outlines were only loosely coordinated within each area in order to leave authors free to invent new approaches. In point of fact, however, a number of monographs do relate to others in complementary ways, a result of their authors' close, informal interaction.

Because of stringent time limitations, few of the monographs have been completed, and none has been extensively rewritten. Indeed, most writers feel that they are barely more than clean first drafts. Yet, because of the highly experimental nature of the undertaking, it is essential that these manuscripts be made available for careful review by other physicists and for trial use with students. Much effort, therefore, has gone into publishing them in a readable format intended to facilitate serious consideration.

So many people have contributed to the project that complete acknowledgement is not possible. The National Science Foundation supported the Conference. The staff of the Commission on College Physics, led by E. Leonard Jossem, and that of the University of Washington physics department, led by Ronald Geballe and Ernest M. Henley, carried the heavy burden of organization. Walter C. Michels, Lyman G. Parratt, and George M. Volkoff read and criticized manuscripts at a critical stage in the writing. Judith Bregman, Edward Gerjuoy, Ernest M. Henley, and Lawrence Wilets read manuscripts editorially. Martha Ellis and Margery Lang did the technical editing; Ann Widditsch supervised the initial typing and assembled the final drafts. James Grunbaum designed the format and, assisted in Seattle by Roselyn Pape, directed the art preparation. Richard A. Mould has helped in all phases of readying manuscripts for the printer. Finally, and crucially, Jay F. Wilson, of the D. Van Nostrand Company, served as Managing Editor. For the hard work and steadfast support of all these persons and many others, I am deeply grateful.

Edward D. Lambe
Chairman, Panel on the
New Instructional Materials
Commission on College Physics

THE CONCEPTUAL FOUNDATIONS OF FLUID MECHANICS

PREFACE

This monograph is incomplete in several respects. Many useful figures and references could be added. Problems (not yet prepared) are necessary to fill out considerations that are only lightly sketched in the text. (Some of the positions at which problems are needed are indicated by the symbol [Problem]).

The discussion may often be heavier than it need be if the content is considered as an end in itself. The material presented here was planned as the first chapter of a textbook on Quantum Mechanics designed for students at an intermediate level (Junior, Senior). The second chapter of the proposed text is to deal with the superposition principle. In this second chapter the usefulness of the many abstractions introduced in the monograph is to be exhibited by employing them for the detailed analysis of microphysical phenomena. Thus to obtain a reasonably thorough understanding of this monograph a study of the as-yet-nonexistent Chapter 2 is required. All this, however, may hardly be worth stating. A theorem of great generality is applicable: "Chapter n of any book in physics cannot be understood until Chapter n + 1 is mastered."

I wish to thank Walter C. Michels and Ernest Henley for their helpful comments and criticisms. I am grateful also to Jack Ludwig and Ralph Caplan for showing me that it is easier to read English than Academese; the numerous changes they suggested greatly improved the style of the monograph. My thanks are due also to the officers of the Commission on College Physics and the University of Washington for their support and assistance during the pleasant, stimulating and productive months of the "Writing Conference."

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"There is no end to our searchings:...
No generous mind stops within itself.
Its pursuits are without limit; its
food is wonder, the chase, ambiguity."

—Montaigne

1 THE FAILURE OF CLASSICAL THEORY

> "For what, by nature and by training,
> We loved, has little strength remaining."
>
> —W. H. Auden

From the time of Galileo and Newton to the close of the nineteenth century, classical physics had met with no experiences that challenged its fundamental concepts. There were, of course, many problems, but their solutions were seen as possible within the boundaries of existing theory. Between 1895 and 1925, penetration into the realm of microphysics - the physics of the elementary constituents of matter and radiation - uncovered phenomena that stubbornly resisted interpretation within the classical conceptual framework. The conflict, in this period, between theory and microphysical observation rocked the very foundations on which classical theory had been built. It was found that to build a successful theory of microphysical processes - the quantum mechanics - it was necessary to eliminate several classical assumptions that had gone unquestioned for centuries.

The character of the great difficulties that classical theory faced may be appreciated from the following brief review.[1] In classical mechanics the properties with which we normally deal (such as energy and angular momentum) are continuously variable. Experiments such as those of Franck and Hertz and Stern and Gerlach indicate, however, that some of the dynamical properties that are continuously variable in classical theory are limited to discrete sets of values, i.e., are "quantized" in microphysical systems. Thus the internal energy or the magnetic moments of a bound atomic system are not continuously variable but may take on only discrete sets of "possible" values. Similarly, the total energy in any sample of radiation of frequency ν is found to be an integral multiple of the "photon energy" $h\nu$. Such quantized dynamical properties find no comfortable place within classical theoretical structures.

Bohr attempted to graft quantization to classical physics by imposing restrictive principles and thereby to limit the possible motions of an atomic system to a subset of the motions permitted by classical theory. Considerable success was achieved by the application of Bohr's ingenious quantization rules. But the resulting theory was incomplete and internally inconsistent; it failed to answer a number of elementary questions. If an atomic system has only a limited set of motions, and these have quite different energies, how does it change its energy from one to another permitted value in processes of emission or absorption of radiation - or in collisions with other material systems - without taking on intermediate values? Does the photon, in an emission process, simply spring into existence with a simultaneous and discontinuous change in the energy of an atom? If so, what determines when this disruptive event occurs? And so on. The marriage of classical mechancis to restrictive principles for the purpose of producing quantization was forced, unstable, and, from the beginning, quarrelsome.

---

[1] It is assumed that the reader has studied the following topics at a level of treatment maintained in strong courses in general physics. The topics are listed in the order of their importance for the work of this monograph.

(a) Photoelectric and Compton Effects; experiments and photon interpretation.
(b) Classical diffraction and interference phenomena; experiments and interpretation by Huyghen's principle.
(c) Rutherford model of atoms and Bohr theory of the hydrogen atom (circular orbits).
(d) Electron beam experiments for the measurement of electron charge and mass.
(e) Franck, Hertz and Stern, Gerlach experiments; interpretation in terms of quantization.
(f) Elementary features of radioactive decay.
(g) Polarization of radiation.
(h) Blackbody radiation.

(Quantization and the constant h first appeared in physics in Planck's treatment of blackbody radiation. Much can be learned from an analysis of different physical interpretations of Planck's theory. Unfortunately, the background required for an understanding of such an analysis is likely not to be available to the students for which this monograph.is designed.

Several references, where treatments of the forefoing topics may be found, are listed at the end of this chapter.

Electromagnetism had its own troubles. It provided no home for the photon; consequently, it could give no description of the phenomena associated with the photoelectric and Compton effects [1,2,3,4,5,6,7,8;Note]. The predictions of electromagnetic theory were in conflict with Rutherford's model of the atom in which electrons are assumed to move in orbits about a central nucleus; radiation from the accelerated motion of an orbiting electron would drain energy rapidly from the atom and the electron would spiral down into the nucleus in a time of the order of 10^{-12} seconds! Both electromagnetism and mechanics were in fundamental conflict with the remarkable stability of atomic properties. How is it possible, for example, for atoms to maintain fixed properties indefinitely, despite the numerous collisions they suffer because of thermal agitation?

Classical theory never quite learned to get along with either the quantization of radiation in photons, or the quantization of various properties of material systems, although it managed a somewhat uneasy relationship for a while. Toward the so-called wave-particle dualism, however, classical theory showed an enduring antipathy. Since early in the nineteenth century when the diffraction properties of radiation were extensively observed, it was supposed that light (later recognized as electromagnetic radiation) was propagated as a transverse wave. The long argument between particle and wave theories which had gone on since Newton's time (Newton himself advocated a particle theory) seemed to be settled by the phenomenon of diffraction for which only the wave theory had a convincing explanation. Maxwell's highly successful electromagnetic theory of radiation served to deepen the belief that radiation must be regarded as a wave phenomenon. But it seemed to be impossible to explain the photoelectric and Compton effects by means of a wave theory. Indeed these effects appeared to be understood more easily on the assumption that radiation consists of entities (photons) that are more like particles than waves.

When the study of the emission, absorption, and scattering of radiation by atoms and electrons made clear the significance of the photon characteristics of radiation, physicists were faced with a disturbing dilemma. If radiation consists of photons, how can diffraction be explained? If the diffraction phenomena indicate that light is a wave, how can the particlelike properties of radiation (photoelectric and Compton effects) be understood? Oddly, the full impact of the issues raised by the existence of both wavelike and particlelike properties of radiation was not felt until precisely the same problem appeared, quite unexpectedly, in the study of electron properties. From the time of its discovery at the end of the nineteenth century, the electron had been considered to be a very good approximation to a Newtonian point particle, and many observations were, of course, consistent with this assumption. But when it was discovered that in the transmission of a beam of electrons through a crystal, diffraction patterns of the same character as those previously observed with x rays were produced, it became clear that electrons were not to be so sharply differentiated from radiant energy, as had previously been supposed [1, Chap. 5; 5, Chap. 6; 8, Chap 4; 15, 16]. The need for a consistent theory that could comprehend the so-called dualistic (wave-particle) behavior of electrons and radiation became painfully obvious.

The diffraction of electrons and the particlelike properties of radiation indicated that far more than mere patchwork would be required to construct a consistent theory of microphysical phenomena. Fortunately, at almost the same time that the electron diffraction was observed, Heisenberg, Schrödinger, Dirac and others were groping toward the formulation of a remarkably successful theory - the quantum mechanics - to replace the now badly mauled classical theories of matter and radiation.

It took many years of observation and thought to realize that certain of the tacit assumptions of classical physics, though entirely consistent with experience in "macrophysics," are simply wrong in the microphysical realm. To obtain a consistent theory of microphysical systems it was necessary to make revolutionary changes in several of the assumptions on which the structure of classical theory is based. The object of this monograph is not to recount yet again the failure of classical theory but to describe the conceptual shift that is required for an understanding of the behavior of the elementary constituents of matter and radiation.[2]

---

[2] An epitome of the discussion to follow is presented in Subsection 9.1. The reader may find it helpful to use this epitome as a guide to the structure of the monograph.

2 CONSEQUENCES OF A MISTRUST OF THEORY

> **"Prohibit sharply the rehearsed response."**
>
> **—W. H. Auden**

Observation on microphysical systems are generally heavily dependent on theory. We cannot "see" electrons or photons in anything like the same sense that we can "see" baseballs. Properties of microsystems are usually "observed" by interactions with complex macroscopic apparatus that produce scintillations on a screen, clicks in a counter, vapor droplets in a cloud chamber, or readings on various kinds of meters. Such raw observations are interpreted by chains of theoretical argument; therefore, if the applicability of the theory employed for analysis is uncertain, the interpretations correspondingly become uncertain.

Suppose, for example, that we find that a certain sample of radiation, when passed through a diffraction grating, is deflected on passage by an angle, θ. The observation of the angle of deflection is generally interpreted as a measurement of the wavelength, λ, of the radiation by the law [9,10,11]

$$\lambda = d \sin\theta, \qquad (2.1)$$

where d is the spacing between the lines of the grating. Also, since radiation in free space has the velocity c (velocity of light) the deflection is further interpreted as a measurement of the frequency, ν, by

$$\nu = \frac{c}{\lambda}. \qquad (2.2)$$

However, we have not measured the distance between crests (say) on a wave form in space nor has the number of oscillations per second of the wave field been counted at a point. We have instead used the theory that the incident radiation is a wave phenomenon; from the theory together with the measurement of the angle of deflection, θ, and the slit separation, d, the wavelength (1) and frequency (2) of the radiation are deduced.

But it is clear from the photoelectric and Compton effects, as well as from a variety of other observations, that photons are not correctly described by a wave theory. Since it is precisely the wave theory that permits us to infer a wavelength from a measurement of θ, its application in this instance is at least doubtful. Whatever it measures, the deflection through θ of the incident radiation tells us something quite definite about the radiation. From a knowledge of θ and d, we can predict the energy of the electrons released by the photoelectric effect. Thus it is known from experiment that the energy transfer, ΔE, per photon, is

$$\Delta E = hc/d \sin\theta \qquad (2.3)$$

(which is just $h\nu$, of course, with ν given by Eqs. (2.1), (2.2)). Also, the characteristics of the diffraction patterns produced by crystals, slits, etc., vary smoothly with the angle θ through which the radiation is deflected by the grating. The deflection θ tells us something very important about the radiation, but unless a correct theory justifies the association, we cannot assert that the deflection measures a wavelength, or a frequency. Of course, it is possible that the wavelength and frequency as determined by Eqs. (2.1), (2.2) have significance in relation to a theory which does correctly describe the behavior of photons. But at this prequantum mechanics stage of our discussion, in which we recognize the failure of the wave theory of radiation and have as yet no adequate theory with which to replace it, we are certainly free to question whether the numbers λ, ν, derived from θ by Eqs. (2.1) and (2.2) have anything whatever to do with the physical properties of radiation.

Consider a second example: Early in this century, Rutherford recognized that the distribution of the charges within atoms could be explored ("observed") through the study of the scattering of energetic α particles by matter. If it is assumed that the forces that deflect an α particle when passing an atom are electrostatic, the character of the scattering by matter is determined by the arrangement of the charges within the atoms of which the scatterer is constructed. Geiger and Marsden (in Rutherford's laboratory) investigated the general characteristics of α particle scattering by means of an arrangement like that shown schematically in Fig. 2.1. The scatterers they used consisted of sheets of heavy metals which were thin enough to permit most of the incident α particle to penetrate the scatterer without sensible deflection. The observations consisted in counting at various angles relative to the direction of the incident beam the number of scintillations produced on a zinc sulphide screen by the arrival of scattered α particles. To Rutherford's considerable surprise, Geiger and Marsden discovered that a small, but far from negligible, fraction of the α particles were scattered through angles larger than 90°. Analysis of the experiments indicated that these large angle scattering events could not have arisen from a number of successive small angle deflections; apparently an energetic α particle (with millions of electron volts of kinetic energy) could be deflected through a large angle as the result of a collision with a single atom of the scattering material.

Rutherford realized that the observations of Geiger and Marsden could not be explained by the atom models then under consideration. In 1911, he proposed a new model [12] - the now well-known nuclear or planetary model - which provided a source for the strong forces required to produce the large deflections suffered by α particles in passing through matter. It was assumed that the positive charge required to balance the negative charges of the planetary electrons of the atom is concentrated in a nuclear core of diameter several orders of magnitude smaller than the diameter of the atom and that the nuclear core contains essentially the entire atomic mass; the electrons were assumed to move in orbits about the core under the influence of the attractive forces between the positive nuclear core and the negative electrons.

With this model, Rutherford calculated the form of the distribution of the scattered α particles that would result if real atoms corresponded to his model. Geiger and Marsden[13] undertook to check his predictions by a pains-taking quantitative study of the scattering of α particles of various energies by sheets of various heavy metals. The agreement of experiment with theory left nothing to be desired. Effectively it had been "observed" that atoms have a structure

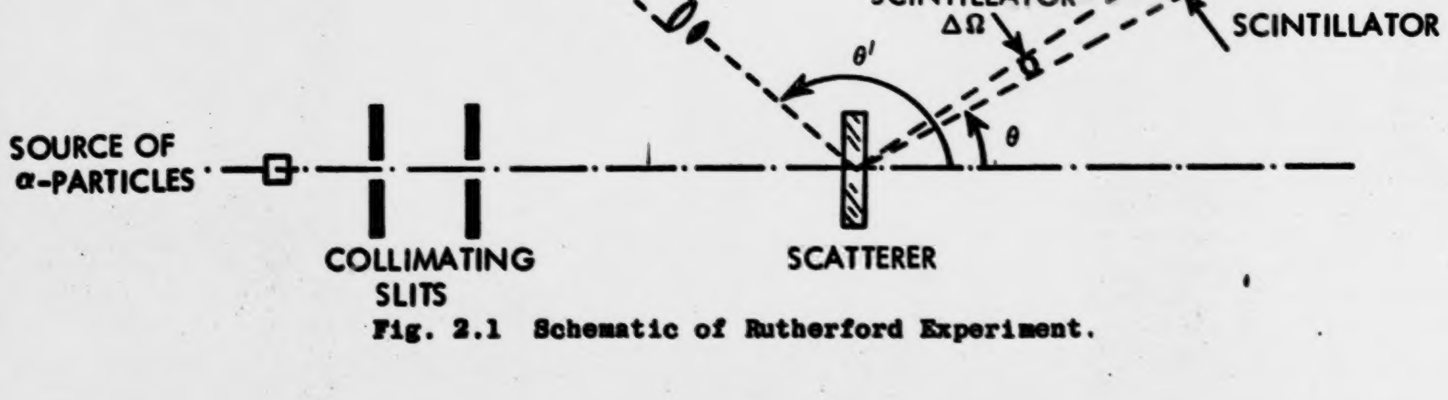

Fig. 2.1 Schematic of Rutherford Experiment.

like that of the model Rutherford created. Despite the many difficulties posed by the model (e.g., in their motions about the nucleus the electrons must radiate energy - according to electromagnetic theory - and spiral very rapidly down into the nucleus), it was clearly necessary to take it seriously. Its acceptance was complete when the atomic theory of Bohr, which is based on Rutherford's model, proved so fruitful. (Bohr's first paper and the results of the work of Geiger and Marsden that verified Rutherford's predictions were both published in 1913). The small, positively charged, massy nuclear core first imagined by Rutherford remains today at the center of theories of atomic structure.

Naturally Rutherford assumed, in his calculations, that the motion resulting from a collision of an α particle with an atom could be described by Newtonian mechanics; his model specified the form of the force law to be used in the classical equations of motion. However, since experience in microphysics indicates that it is doubtful whether classical conceptions are applicable to atomic processes, the validity of Rutherford's "observation" of the nuclear core of the atom is open to serious question. Perhaps a model of the atom very different from that imagined by Rutherford together with a correct theory of microphysical processes would duplicate his predictions. Such coincidences are no doubt rare but they are not unknown. As it happens the quantum mechanics - a successful theory of microphysics - together with the physical assumptions introduced by Rutherford leads to exactly the same scattering distributions as those obtained from classical theory.[3] We are being led away from the point, however, by the intrinsic interest of the story. For our present purposes it is essential to note only the strong interplay between theory and experiment in Rutherford's "observation" of the nuclear core of the atom.

The point needs no further reiteration. Other examples abound. What we conclude is that in the realm of microphysics, where measurement is necessarily indirect, our concepts derive from a close interweaving of theory with observation. In this combination, the role of theory grows as investigation goes on. Ultimately theory becomes so familiar that we hardly realize its importance in the interpretation of observation; the deflection of radiation by an angle, θ, on passing through a grating is almost automatically interpreted as a measurement of wavelength and the role of theory in the measurement is almost forgotten. When theory fails, however, the familiar connections between its constructs and what is observed are broken. We must then return to naked observations and their observed interrelations, and try to build from them new, and successful, theoretical structures.

The task is enormously difficult. Thinking, without preconceptions, is probably impossible. Our very language has been conditioned by our experiences with macroscopic phenomena and, consequently, may be ill adapted to the needs of microphysics. The sheer task of describing raw observation is awkward and toilsome. Thus we say we are "observing" the diffraction of electrons when in actuality we are reading meters connected to certain pieces of physical apparatus and measuring the intensity pattern on a photographic plate. But the very use of the term "diffraction" commits us to thinking in terms of a wave theory and conjures up images not at all connected with what is observed.

---

[3] This most happy coincidence between the results of quantum mechanics and classical mechanics for the scattering distribution occurs only for the inverse square law of force. For other force laws, the predictions of the two theories may differ considerably. It is amusing to speculate on how long the development of atomic physics might have been held up if the two theories had led to markedly different scattering distributions for the inverse square law of force.

This leads into a bind: if we stick to the language of pure observation, discussion becomes ponderous and inordinately wearisome; if we proceed rapidly through the introduction of theoretical constructs, the risk of misconception and error is large. In spite of its dangers the latter course must be chosen.[4]

In our effort to shake loose from the grip of classical conceptions, we have probably overemphasized the extent of the failure of classical theory. Electromagnetic theory certainly correlates a vast set of observations on the properties of radiation. Many characteristics of atomic systems - for example, the normal Zeeman effect - may be interpreted successfully by means of Newtonian mechanics. The Bohr theory of the hydrogen atom, a classical theory to which a restrictive rule is added, enjoyed many successes. Though many fundamental classical conceptions are inadequate for the needs of microphysics, it is beyond question that a correct theory of entities on the atomic scale (electrons, atoms, photons, etc.), must overlap classical descriptions in some way.

---

[4] We shall use derived quantities such as the λ and ν in Eqs. (2.1), (2.2) but try at the same to keep in mind the experimental and mathematical procedures by which these quantities are obtained. One must try to avoid thinking of λ, ν as the wavelength and frequency of some physical wave. These symbols refer to quantities which are obtained from raw observations by easy computations; they are used because they help to express, in a simple way, observed relations among phenomena.

3 PROPERTIES OF ELECTRONS, PHOTONS; THE DE BROGLIE RELATIONS

> "Does he wear a turban, a fez, or a hat;
> Does he sleep on a mattress, a bed, or a mat?"
> —Edward Lear

It is not our purpose to review all the observations on microphysical entities in order to expose those interpretations that are tainted by unwarranted use of classical concepts; most of such a review would teach us little. Many conclusions about the microphysical realm are consistent with so great a variety of experiments, or are so largely independent of theoretical interpretation, that they are hardly subject to doubt.

It is obvious that the entities named electrons play an important role in atomic constitution. We know that the charge, e, and rest mass, m, of the electron[5,8] are

$$e \cong -4.8\cdot 10^{-10}\,\text{esu} \cong -1.6\cdot 10^{-19}\ \text{coulombs}$$

$$m \cong 9.1\cdot 10^{-28}\,\text{g}.$$

The energy, E, for free electrons is related to the momentum, p, by

$$E = p^2/2m$$

($E \ll mc^2$; nonrelativistic approximation) (3.1a)

$$E = c[p^2+(mc)^2]^{\frac{1}{2}} \quad \text{(relativistic)} \quad (3.1b)$$

The velocity, v, of free electrons is found to be related to the momentum as in classical mechanics:

$$v = p/m \quad (v \ll c) \qquad (3.2)$$

$$v/(1 - v^2/c^2)^{\frac{1}{2}} = p/m \quad \text{(relativistic)}$$

Electron beams are deflected under the influence of laboratory-produced electric and magnetic fields.[5] The deflections may be computed using Newton's second law $\vec{F} = d\vec{p}/dt$ with

$$\vec{F} = e\left[\vec{E} + \frac{\vec{v}\times\vec{H}}{c}\right] \qquad (3.3)$$

where $\vec{E}$, $\vec{H}$ are the electric and magnetic fields at the location of the electron.

Photons may be interpreted as the elementary constituents of radiant energy; they are uncharged and are transmitted in free space with the veolocity of light. On interaction with material systems (electrons, atoms, etc.), they transfer specific quantities of momentum and energy. The relation between these quantities is found to be

$$E = cp. \qquad (3.4)$$

This relation follows from Eq. (3.1b) if we set m = 0; consequently the photon is said to be an entity with zero rest mass.

The possible energy values for photons range over an infinite continuum. It is found that photons to which one may assign a "frequency," ν, have a uniquely correlated energy given by

$$E = h\nu \qquad (3.5a)$$

(where h, of course, is Planck's constant). By the use of (3.4) and the relation $\lambda\nu = c$, we have

$$p = E/c = h\nu/c = h/\lambda.$$[6]

---

[5] The word "laboratory" here means that the experiments are conducted on a macroscopic scale with macroscopic controls. The Newtonian concept of force loses its meaning for electron "motions" on a microscopic scale (see Section 6).

[6] Properties of energy and momentum are attributed to classical electro-magnetic fields. However, the energy and momentum are interpreted as spread continuously over the field. One speaks, in the theory, of energy, momentum densities, and of the flow of energy, momentum across surfaces. In the interaction with matter of a wave of frequency ν (and this quantity has an unambiguous meaning in the theory) energy and momentum are continuously transferred rather than in the lumps $h\nu$, h/λ as in the photon picture.

Photons exhibit properties associated with the term "polarization.[7] The angular momentum carried by circularly polarized photons has been measured and found to be $h/2\pi$.

It is important to recognize that a photon of "frequency," ν, is as indecomposable as an electron. Just as we never find parts of an electron, we never find photons of a given frequency (i.e., photons which are deflected by the same angle θ in a given grating), with a fraction of the energy $h\nu$. All efforts to cut a photon into parts each with the same "frequency" but with fractions of the energy $h\nu$ have failed. Like electrons, photons may suffer energy changes in collision processes (e.g., the Compton effect) but in such changes the "frequency" also alters.

Most of the properties we have discussed for electrons and photons enter into conservation laws. A multitude of observations are consistent with the assumption that the general conservation laws - for energy, momentum, angular momentum, charge - are valid in the microphysical realm, These conservation laws make it possible to give clear meanings to the measurements of conserved quantities, meanings which are independent of the nature of more detailed theory.

3.1 THE DE BROGLIE RELATIONS.

Both electrons and photons produce "diffraction effects" under suitably arranged conditions. The diffraction effects are similar, in some respects, to those produced by classical waves, and may be described, in part, by a classical wave theory. An examination of the diffraction effects indicates that the "wavelength" which must be employed by the classical wave theory to describe the observations is related to the observed momentum of the incident beam of electrons or photons by

$$\lambda = h/p. \qquad (3.6a)$$

We shall have occasion in the next chapter to associate a "frequency," ν, as well as a wavelength, λ, with electrons of momentum p; this "frequency" (which will not be interpreted as the number of oscillations per second of a physical wave) may be related to the electron energy by

$$\nu = E/h. \qquad (3.6b)$$

Thus the relations (3.6a), (3.6b) between E, p, and ν, λ hold for photons and for electrons.

The equations

$$\lambda = h/p \qquad \nu = E/h, \qquad (3.6c)$$

which connect the "particle properties" p, E,[8] with the "wave properties" λ, ν, are generally known as the "de Broglie relations." Some years before the observation of electron diffraction, de Broglie, in his doctoral thesis, suggested that the properties of electrons might be understood better on the assumption that the electron constituted a wave phenomenon of some kind[14]. He assumed that the wavelength and frequency of his "electron waves" are related to the energy and momentum of electrons in exactly the same way as the wavelength and frequency of photons are related to their momentum and energy. De Broglie published his theory in 1924. His conceptions were considered rather fanciful until, about a year later, Schrödinger took them up and extended them into his system of "Wave Mechanics." It was not until 1927 that the phenomenon of electron "diffraction" was observed and the relation $\lambda = h/p$, first suggested for electrons by de Broglie, was confirmed[15,16].

---

[7] Electrons also have properties analogous to the polarization properties of photons; these properties are referred to by the name "spin." We do not wish now to enter into a discussion of the spin properties of electrons.

[8] See footnote 6.

That the de Broglie relations hold for such dissimilar entities as electrons and photons is most remarkable. These relations express compactly the connection between the wavelike properties associated with λ, ν, and the particlelike properties associated with E, p. Note in particular that the connection involves the new constant h. The magnitude of h determines the conditions under which the wavelike properties of electrons will be important (see Sections 5, 6).

The measured value of h is

$$\begin{aligned} h &= 6.62 \cdot 10^{-27} \text{ erg} \cdot \text{sec} \\ &= 4.14 \cdot 10^{-15} \text{ eV} \cdot \text{sec}. \end{aligned}$$

We find from Eq. (3.6c), on setting $p = \sqrt{2mE}$, (for electrons with nonrelativistic energies) that

$$\lambda \cong 12.3\ E^{-1/2}\ [\text{Å}(\text{eV})^{1/2}]. \qquad (3.7)$$

(Å ≡ angstroms; eV ≡ electron volts) Thus, for scattering centers with separations of the order of an angstrom - such spacings occur in crystals - diffraction effects are large for the full range of electron energies below 10^3 eV. For electron energies of the order of 10^6 eV, however, the wavelength is approximately .01Å; hence diffraction effects about objects of atomic dimensions (of the order of an angstrom) will be small and the electrons will appear to behave like particles. For photons the relation $E = h\nu = hc/\lambda$ leads to

$$E \cong 1.2 \cdot 10^4\ \lambda^{-1}\ (\text{eV Å}). \qquad (3.8)$$

For visible radiation with $\lambda \cong 5000$ angstroms, the photon energy is roughly 2 eV. An energy transfer of this magnitude has significant effects for atoms; consequently, the particlelike nature of visible radiation is important in atomic processes. For smaller wavelengths - x rays, γ rays - the particlelike nature of photons becomes more pronounced. If we consider, however, radiation in the microwave region with $\lambda \sim 10^8$ angstroms (= 1 cm) the photon energy is only 10^{-4} eV. In the normal processes of transmission and reception of such radiation the photon character of the radiation would be difficult to observe.[9]

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[9] It is worth recalling that it is not always an easy matter to determine whether a given process which transmits energy is more suitably described as a stream of particles or as a wave process. The debate that began in Newton's time over the character of light went on for several decades. Newton, on failing to detect diffraction effects, favored a particle theory of light and his hunch was backed by most physicists for more than a century. The issue appeared to be settled against Newton's view early in the nineteenth century by the work of Young and Fresnel on diffraction. However, after the photon characteristics of radiation were discovered, it was recognized that the nature of light was not yet fully understood.

4 AN ANALYSIS OF ELECTRON DIFFRACTION

"But whether that be true or no
The Devil any of you know."
—Samuel Butler

Perhaps the most surprising of all the strange phenomena that have been observed in the microphysical realm are those associated with the so-called wave-particle dualism. It is in this area that the failure of classical concepts becomes most acute.

A typical experiment on the diffraction of electrons[8,15,16,17,18] is arranged as shown in Fig. 4.1. An electron source - say a hot filament similar to that used in radio tubes - is put behind a pair of plates in which there are circular holes with diameter of the order of millimeters (a macroscopic dimension). The plates are separated by about 10 cm; between the filament and the first plate a potential difference of the order of several hundred volts is maintained. Some of the electrons that are "boiled" out of the heated filament find their way through the hole in the first plate; in this process the electrons are subject to a fairly uniform accelerating field and gain an energy of (say) 100 electron volts. The electrons that get through the hole in the second plate constitute a well-collimated beam; the number of electrons per second entering this beam may be controlled by a variety of devices. The energy distribution and the geometric characteristics of the beam of electrons that emerge from the accelerating system may be directly tested. Suppose that the experimental arrangement determines the energy to an accuracy of one percent. Since $p = \sqrt{2mE}$, the momentum magnitude is accurate to about one-half percent. The size of a cross section of the beam may be examined at various distances from the exit hole by examining the effects of the beam on a photographic plate. This cross section is

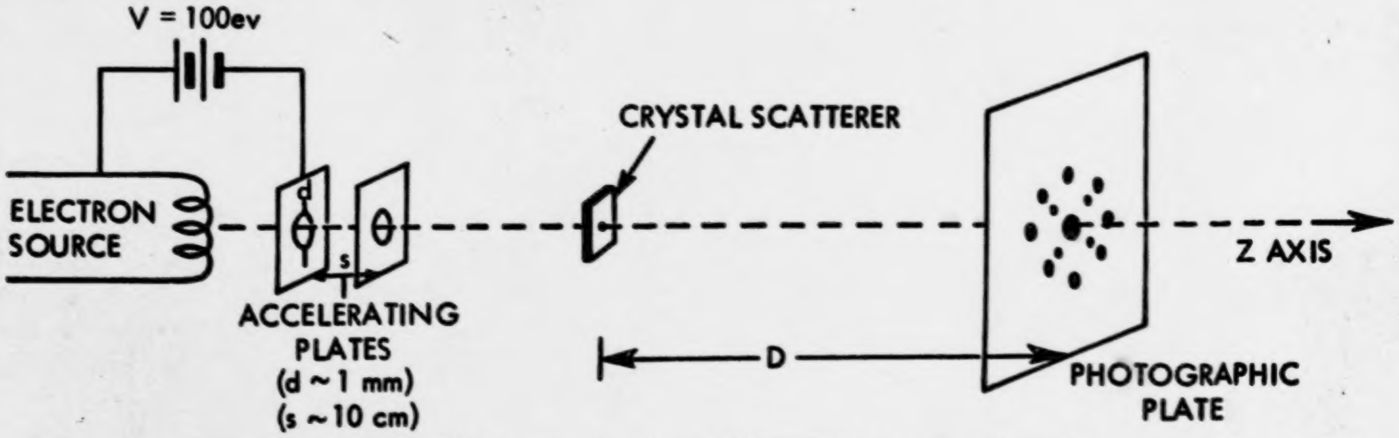

Fig. 4.1 Schematic of electron diffraction experiment.

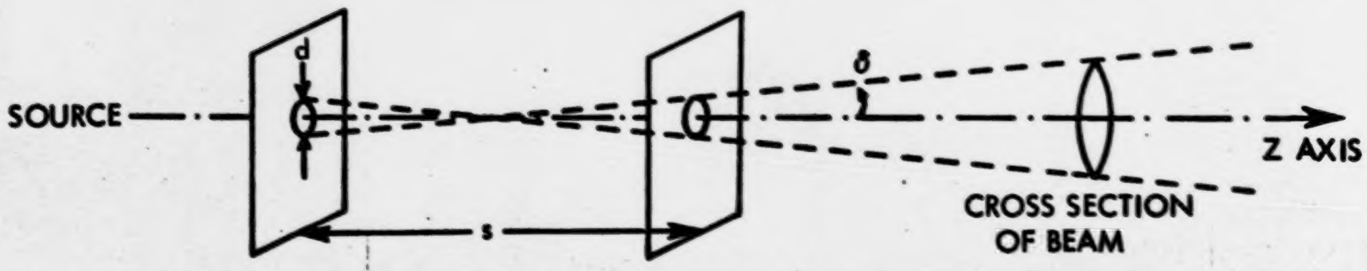

Fig. 4.2 Angular spread of electron beam.

found to vary with distance (to within the accuracy of the observation) as one would expect on the assumption that electrons move on straight line paths (Fig. 4.2). It is easily seen that the angle, δ, of the conical region in which the electrons are found is (in radians) the ratio, d/s, of the hole diameter, d, to the plate separation, s. For the dimensions, $d \sim 1$ mm, $s \sim 10$ cm, this angle is 10^{-3} radians. Thus the transverse component (i.e., the component in the plane perpendicular to the direction of the electron beam) of the momentum of an electron is less than about $10^{-3}\ p_0$ where $p_0 \cong \sqrt{2mE}$ is the average value of the component of the momentum along the direction of the beam (Fig. 4.3).

On the axis of the electron beam produced as described above, a thin section of a crystal is placed. A photographic plate is put at a convenient distance, D, from the crystal along the direction of the electron beam; D may be of the order of ten centimeters. When the filament is hot, electrons emerge from the accelerating system, strike the crystal and suffer deflections. The photographic plate serves as a detector for the location of the electrons in a plane transverse to the beam. A photograph of the pattern that appears on the plate in an experiment of this sort is shown in Fig. 4.4.

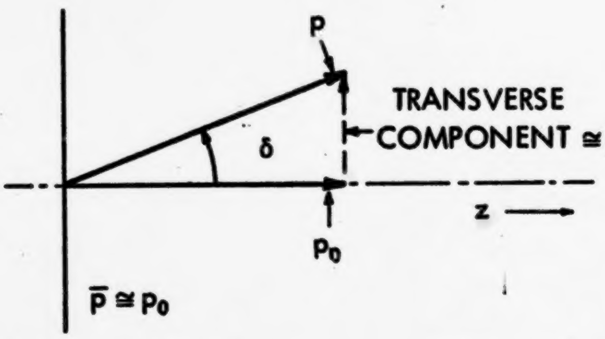

Fig. 4.3 Transverse component of electron momentum.

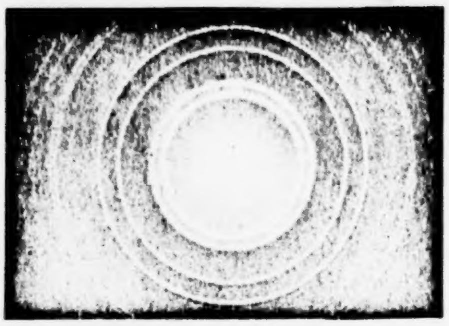

Fig. 4.4 Photograph of the diffraction pattern produced by passing a beam of electrons through a sample of aluminum power. The electron wavelength employed was 0.15 Å. Courtesy, Film Studio, Education Development Center, Inc.

The character of the pattern produced on the photographic plate is independent of the intensity of the incident electron beam; if the number of electrons per second is cut by a fraction f and the time of the exposure of the photographic plate is increased by the factor f^{-1} (so that the total number of electrons recorded on the plate is kept constant), the picture produced is unchanged. The same general pattern is found on the photographic plate if a beam of x rays (photons) of suitable energy is employed in place of the electron beam (Fig. 4.5). The form of the observed pattern depends on the structure of the crystal scatterer employed in the experiment.

If the arrangement and spacings of the atoms within the crystal are known, the pattern observed on the photographic plate can be calculated by assuming that the incident beam may be replaced by a plane wave propagating in the direction of our beam with a wave length given by the de Broglie relation, $h/p_0 \cong h/(2mE)^{\frac{1}{2}}$ where E is the incident energy of the electrons. In the calculation one simply assumes that each atomic site is

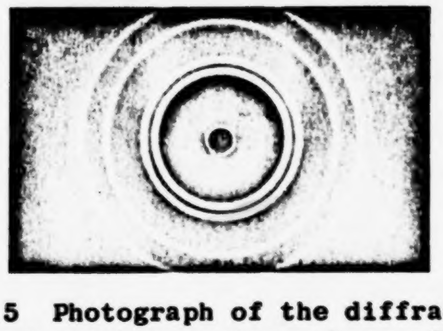

Fig. 4.5 Photograph of the diffraction pattern produced by passing a beam of X-rays (λ = 0.71 Å) through a sample of aluminum powder. Courtesy, Film Studio, Education Development Center, Inc.

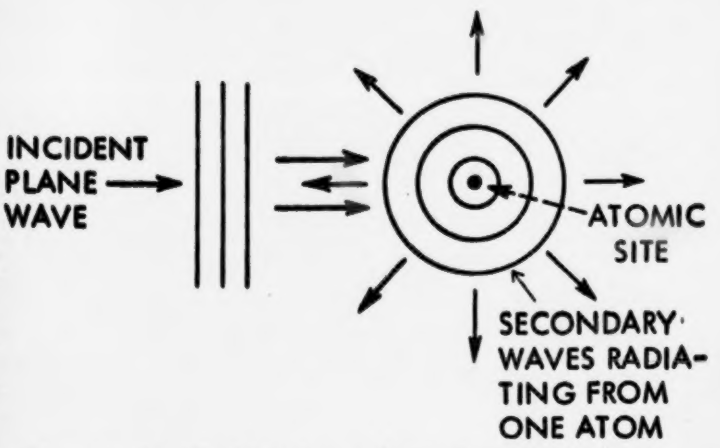

Fig. 4.6 Scattering of an incident wave from an atomic site.

the source of a secondary wave that emerges radially from the atom (Fig. 4.6). It is found that constructive interference of the waves from the numerous atomic sites occurs only at the positions at which the darkening of the plate is observed.

We have described the electron diffraction experiment essentially as it was first performed by Davisson and Germer and by Thomson. For the purposes of the analysis to follow we shall consider the experiment in an idealized form. The incident beam in this experiment is assumed to have well-defined energy and momentum and a cross section such as shown in Fig. 4.7. We replace the crystal by a plate on which narrow slits are cut out. The geometry of these slits and the arrangement of the detector (a photographic plate, a sheet of scintillating material or an electron counter) is shown in Fig. 4.7. An experiment with an arrangement of this kind in which laboratory maufactured slits were used was performed relatively recently by Jönsson[19] who used slits with a width $\tau \cong 0.2$ microns and spacing $\Delta \cong 1.5$ microns. His photographs of the diffraction patterns produced by a single slit and by a double slit are shown in Figs. 4.8 and 4.9. The observed patterns are similar to the patterns obtained by the diffraction of light[9,10,11]. The observations are deducible from a classical wave theory in which the wavelength employed is $\lambda = h/p$ where p is the momentum of the incident electrons. For a single slit the first minimum occurs at a position, P, (see Fig. 4.10) such that the distances from the two edges of the slit to P differ by a wavelength. It is easily seen from the figure that $\theta \approx \lambda/\tau$ (for small θ) so that the distance OP $\approx D(\lambda/\tau)$. (The intensity pattern predicted by the wave theory is shown in Fig. 4.10.) For two slits a succession of maxima and minima are produced as a result of the interference of the waves emerging from the separate slits. If the slit widths are small compared with the separation, Δ, the slits may be

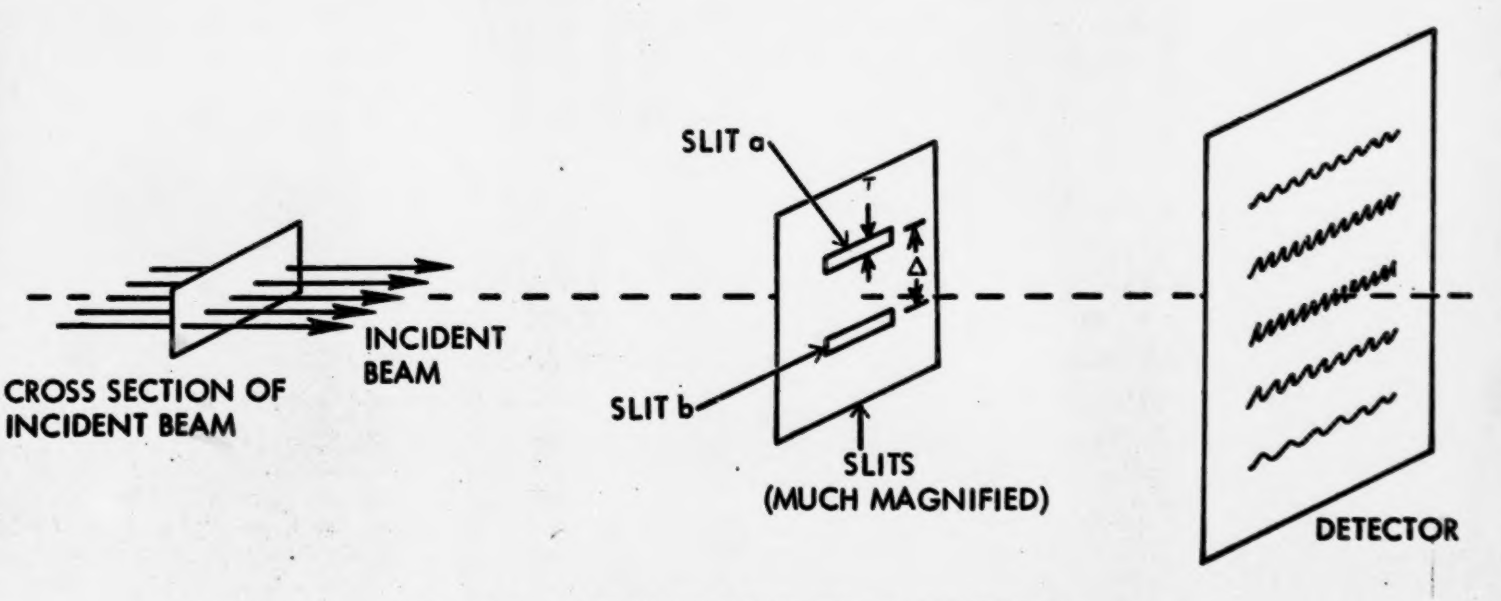

Fig. 4.7 Idealized diffraction experiment.

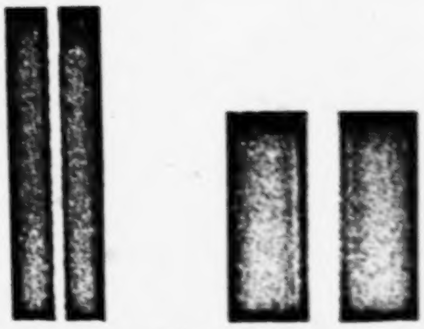

Fig. 4.8 Photographs of single slit and resulting diffraction pattern in Jönsson's experiment. From Jönnson, Zeitschrift für Physik, 161, 454 (1961).

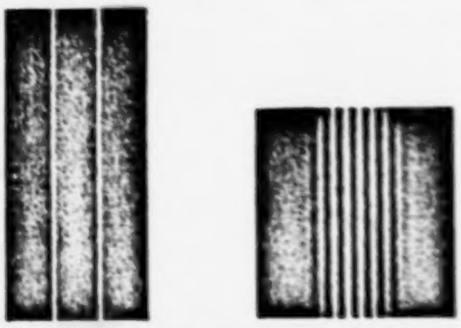

Fig. 4.9 Photographs of double slit and resulting diffraction pattern in Jönsson's experiment. From Jönsson, Zeitschrift für Physik, 161, 454 (1961).

treated as line sources. The position of the first maximum of the intensity (off the central maximum) is at a point Q (Fig. 4.11); the distances from the two slits to Q differ by a wavelength. Thus $OQ \approx D(\lambda/\Delta)$. If Δ is of the order of $5T$ several maxima and minima are obtained in the two-slit pattern covering the region of the central maximum obtained with a single slit.

Remember that the structure of the patterns produced in these experiments is independent of the intensity of the incident beam. Therefore, a weak beam with the number of incident electrons per second such that the probability of finding two electrons within, say, a centimeter of each side of the slits is of the order of 10^{-3} may be used. In other words the diffraction experiments may be performed with one electron at a time in the neighborhood of the slits.

So far there seems to be no argument against considering the elec-

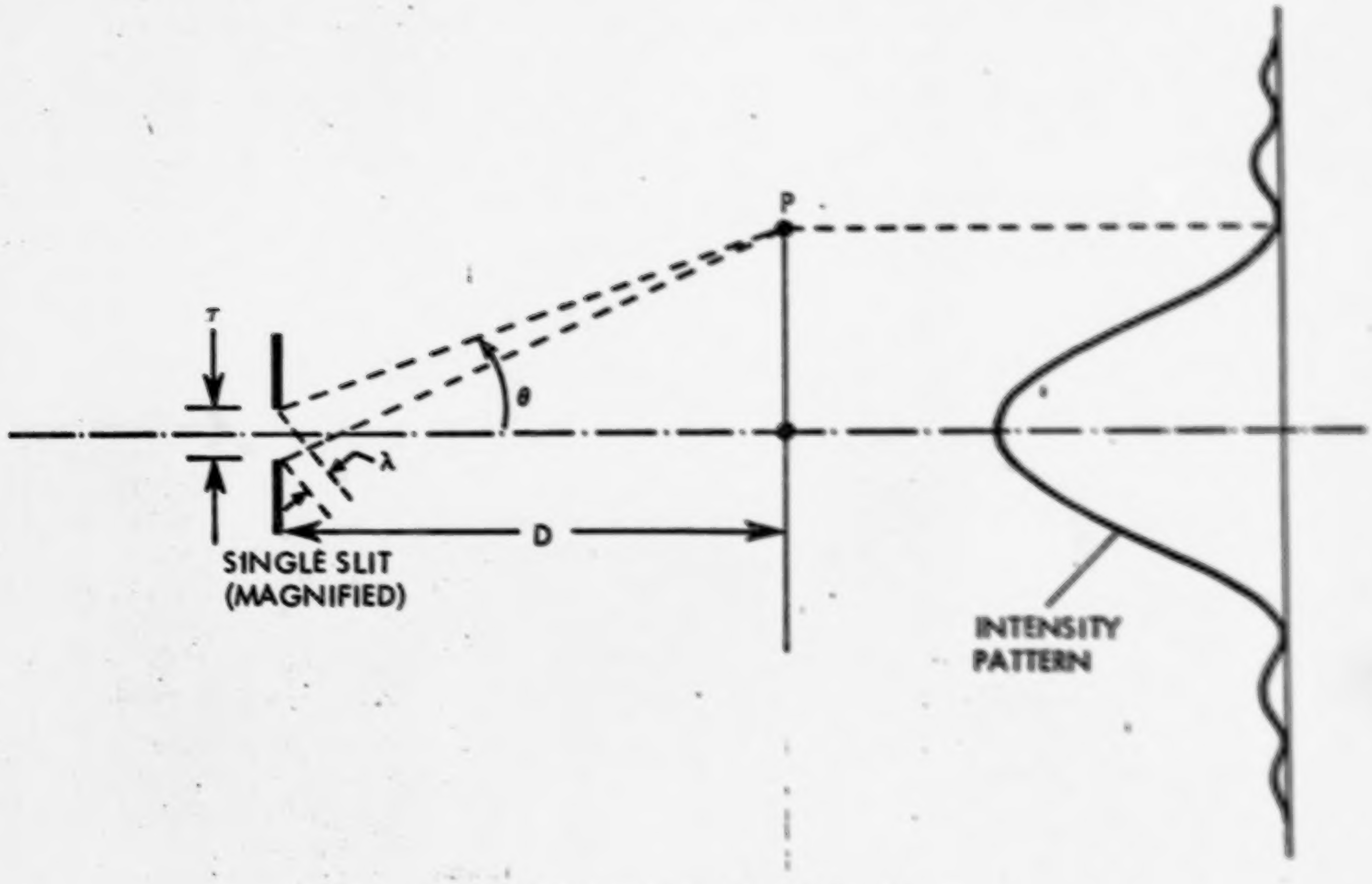

Fig. 4.10 Single slit diffraction.

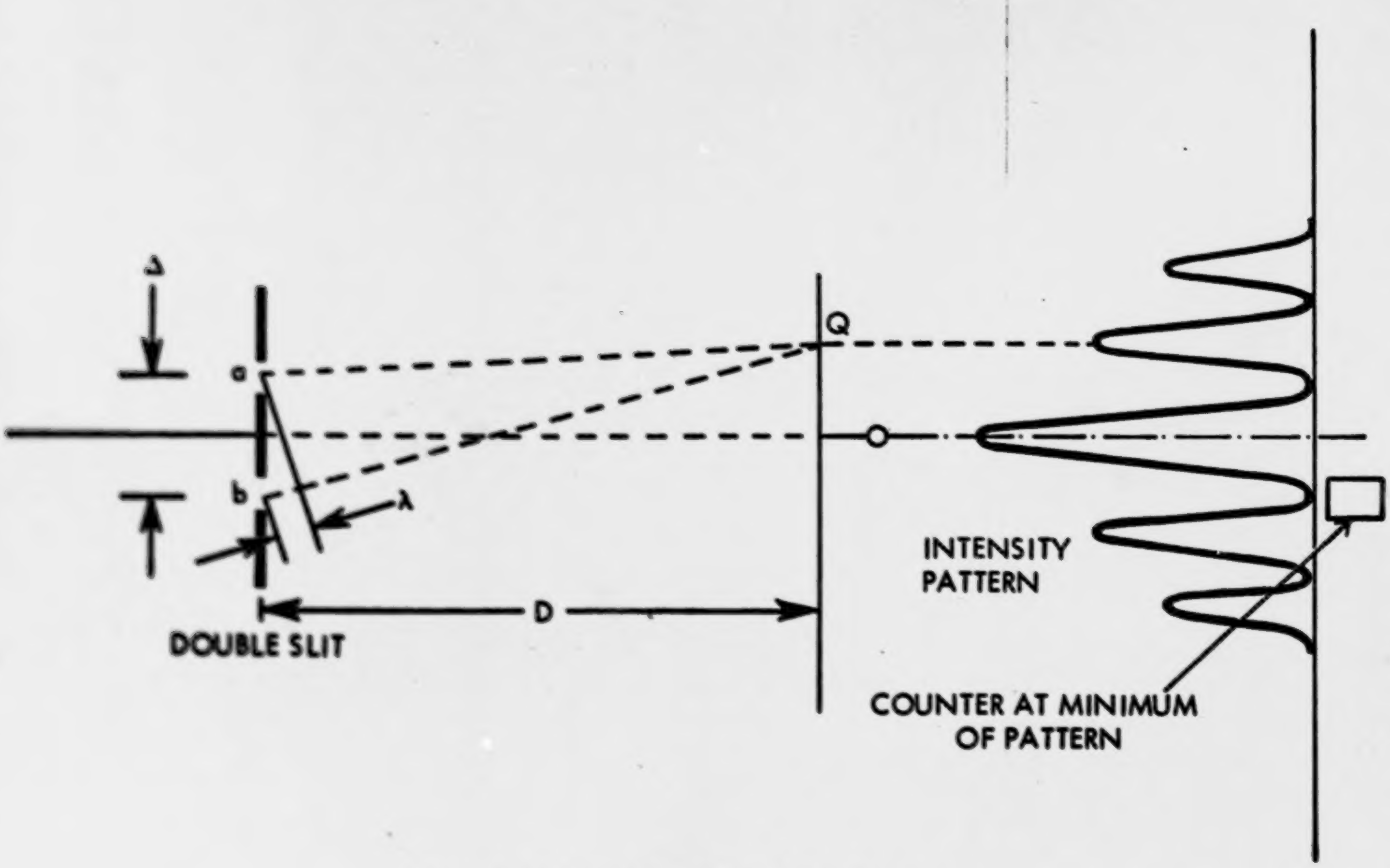

Fig. 4.11 Double slit interference.

tron as a manifestation of some sort of wave motion. However, the model breaks down completely as soon as we consider what happens at the detector. If a scintillating screen is used instead of a photographic plate each electron produces at the screen a well-localized scintillation[10] while successive electrons appear to fall more or less at random on the screen. Thus *an electron does not produce a diffraction pattern*: The patterns are formed from the distribution on the detector of a large number of electrons. The intensity at some position on the photographic plate is proportional to the number of electrons that are deflected into this position. *The diffraction pattern, in short, is not the property of one electron but of an ensemble of similarly prepared electrons*. Classical wave theory does not predict this feature of our observations.

When we take into account the indivisibility of an electron, the structure of the two-slit pattern presents a paradox. Since the electron cannot be divided, it (presumably) must pass through one or the other of the two slits. Surely the open character of the slit through which the electron does *not* pass cannot affect the path the electron takes in getting to the detector.[11] We conclude, therefore, that the two-slit pattern should be exactly the same as that which would be produced if we exposed a photographic plate for a time T with only the "a" slit open and then, for

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[10] Recall that in an observation of an electron we always find a whole electron or none at all; a fraction of the electron charge is never observed.

[11] The diffraction pattern is wholly independent of the nature of the "opaque" materials which form the slit, or of the distribution of the materials in the opaque regions.

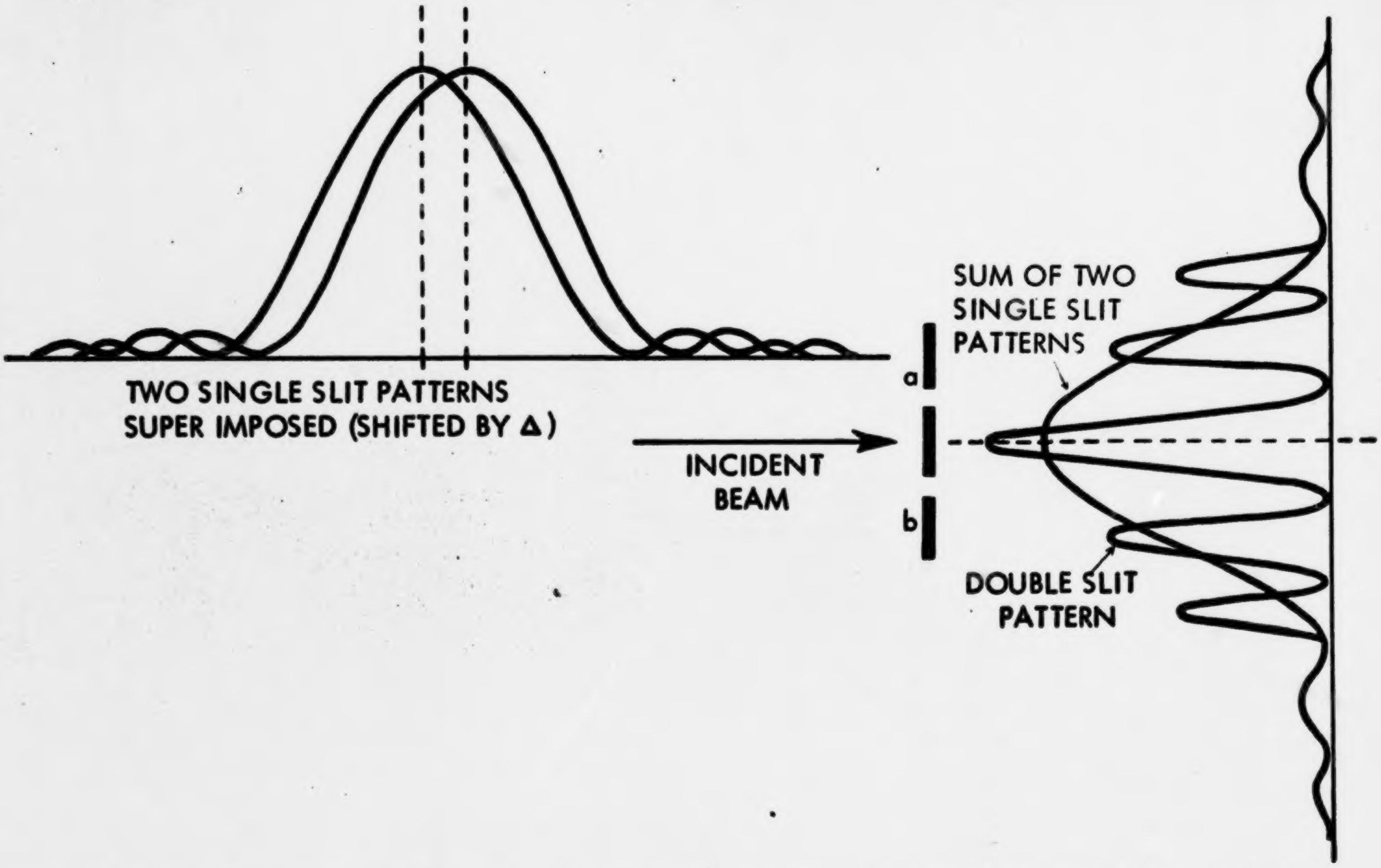

Fig. 4.12 Comparison of sum of two single slit patterns with double slit pattern.

the same period, with only the "b" slit open. The pattern observed with both slits open, however, is very different from that obtained by superposing two single-slit patterns (see Fig. 4.12). The difference between the result predicted (on the assumption that each electron goes through one or the other of the slits) and the pattern observed may be shown rather dramatically by putting a counter at the first minimum of the two slit-pattern (Fig. 4.11). With both slits open, few counts per second are recorded; if, however, one of the slits is closed the counting rate increases! On the basis of the foregoing argument, the closing of a slit could never increase the counting rate for any position of the counter. Apparently something very like an interference effect occurs; to obtain interference we must have influences simultaneously from both slits, i.e., the electron must somehow get through both slits. But presumably the electron is an indivisible entity and cannot go through two separated slits. The paradox appears to be unshakable.

It could be considered ridiculous simply to guess at what is going on as the electron passes a double slit. Under the conditions of the diffraction experiment, different electrons reach the detecting screen at very different points; we have, as yet, no idea, of what determines where any particular electron that gets through the slits will be found at the detector. Why don't we arrange to observe the passage of the electron? By suitable observations we could determine definitely whether an electron can or cannot somehow get through both slits and how the electrons which enter the slits along different paths are deflected.

Sadly, it is impossible to observe the precise path of an electron[12]

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[12] Approximate electron paths are observed in cloud and bubble chamber pictures. The limitations on the observability of electron path is explained below and in Section 6.

(as Section 6 will show). Suppose, however, we try for the moment something a little less ambitious. In the experimental arrangement of Fig. 4.7, an incident beam with a width, t, of the order of a millimeter is used whereas the slit width, τ, and the slit separation, d, are orders of magnitude smaller; the incident beam simply blankets the slits. Why not use a beam narrow by comparison with the width τ and arrange to move it across the slits? We could then have electrons pass an edge or the middle of one of the slits and see where such well-aimed electrons reach the detector. No doubt the reader has already noticed the errors on which the foregoing suggestion is based. If we try to aim the electrons accurately by the use of narrow slits new diffraction effects are introduced. A pair of very narrow aligned slits does not determine a correspondingly narrow beam for the electrons. If the openings in the accelerating plates are narrow and a detecting screen is placed at the position of the diffracting slits, we do not find a geometric image of the slit on the second accelerator plate just as we do not find geometric images of the slits a, b in our diffraction experiment. The diffraction phenomenon severely limits our ability to aim electrons.

Indeed we employ openings in the accelerator plates with relatively large dimensions ($\approx$ 1 mm) in order to avoid introducing significant diffraction effects in the incident beam. For electrons with energies of the order of volts and a slit width of about a millimeter, the angle through which the emerging electrons are spread by diffraction effects is about 10^{-6} radians ($\cong \lambda/t$) (Fig. 4.7) a small angle compared with the spread, δ, (see Fig. 4.2) arising from the geometry of the slits.

Let's consider a different experiment. We noted that the interference phenomenon could not be understood at all if it was assumed that the electron passed either slit a, or slit b, but never both. The issue raised by this consideration may be investigated. Immediately behind slits a and b, place detectors A, B capable of "observing" the passage of an electron. The detectors must be such that if slit a is closed detector A never responds, while if a is open and b closed each electron that passes is detected. Obviously the same must be true if we interchange a, A with b, B in the foregoing remark. (An example of a pair of detectors with these properties is considered in subsection 5.1.) We assume again that the incident beam is so weak that two electrons are never simultaneously in the immediate neighborhood of the slits. If in the course of the passage of the electron beam, the A, B detectors never (or rarely) respond "simultaneously" - i.e., within an interval short by comparison with the average interval between the arrival of successive electrons in the beam - it must be concluded that each electron either gets through a or through b. Simultaneous passage of an electron through both slits would be indicated by a response of both A and B to the passage of a single electron.

This experiment - the diffraction experiment with the A, B detectors - is possible in principle, but forbiddingly difficult in practice. It has never been performed. However, a wide range of experience indicates that an electron is never detected simultaneously at two separate positions. Suppose then that in this "Gedanken experiment" (or pencil and paper experiment) the detectors A, B never respond simultaneously, i.e., the electrons are indeed found either behind a or behind b. Under these circumstances surely the pattern observed on the photographic plate must consist of the simple superposition of two single-slit patterns.

Fortunately, for our peace of mind, there is good reason to believe that in the experiment using detectors, the pattern that would be observed (if the experiment could be done) is such a superposition. The process of observing position - as the next subsection will show - cannot be performed without influencing the electron. The experiment with the detectors is physically different from the experiment without them. The analysis of subsection 5.1 will show that if detectors are arranged to change the electron's properties as little as is possible (consistent with the requirement that a response of A cannot occur if slit a is closed) the pattern produced at the photographic plate becomes roughly a simple superposition of two single-slit patterns (without detectors). In the experiment with detectors, then, the results are consistent with expectation.

Notice that the observation of the electron passage does not help us to understand the normal two-slit pattern. The presence of the detectors radically alters the distribution produced on the photographic plate. To understand the ordinary two-slit diffraction we are forced to the peculiar assumption that in the normal experiment the electrons somehow are influenced by both slits (i.e., get through both slits) despite the fact that a determination of location always discovers the electron at one or the other of the two slits. Reasons for believing that this idea constitutes more than a simple confession of complete defeat will be advanced in Sections 6 and 7.

Effects similar to those described above arise frequently in the microphysical realm. With a certain experimental arrangement an interesting phenomenon is observed. If, however, we attempt to examine the processes that give rise to the phenomenon of interest, we discover that the examination alters the previous observations radically.[13] Nature seems determined to prevent us from discovering some of her secrets.

We don't appear to have come near our destination - an understanding of the two-slit interference experiment - but perhaps the scenery along the way has been interesting. Note in particular the following two very remarkable features of the diffraction phenomena:

1. The diffraction pattern is not a property of a single electron but rather the property of a large collection (ensemble) of electrons.
2. The laws of nature conspire to prevent us from examining the details of the processes that occur in the diffraction experiment. We cannot arrange, for example, both to produce the two-slit pattern and to know with certainty how each electron gets through the slits.

More can be learned from an examination of the one-slit diffraction pattern (Fig. 4.10). The component along the incident direction of an electron is not altered in its passage through the slit. By the arrangement pictured in Fig. 4.13, it can be shown that the electrons which reach the detector are deflected on passing the diffracting slit by an angle θ (of course, the openings in the collimating plates must be large - say about a millimeter wide - so as not to produce further diffraction). The y component of the momentum after deflection (Fig. 4.14) is $p_0 \tan\theta \geq p_0 \sin\theta$ where p_0 is the incident

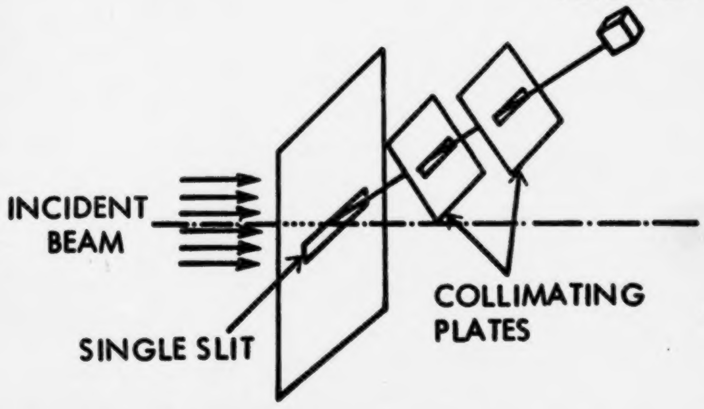

Fig. 4.13 Change of electron momentum on passing slit.

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[13] College students should not be surprised by such effects. They often complain (perhaps justly) that frequent tests of their knowledge tend to destroy the knowledge they may have acquired.

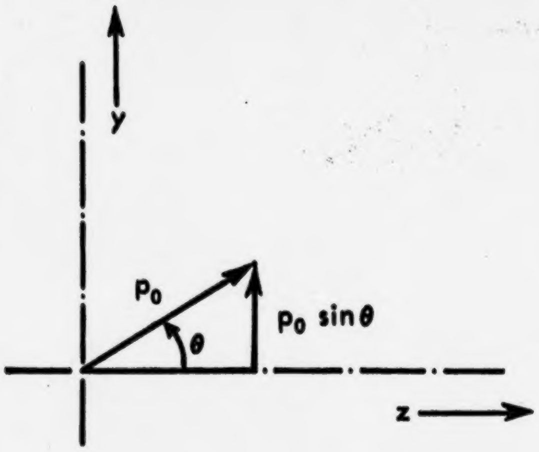

Fig. 4.14 The y-component of momentum after passing slit.

electron momentum. From previous work (see Fig. 4.10) we know that the bulk of the deflections experienced by an ensemble of electrons in passing the slit lie between the limits, $\pm\theta'$, where θ' is given by

$$\sin\theta' = \lambda/\tau,$$

($\lambda = h/p_0$ and τ is the slit width). Hence the y components of the electrons after passage range roughly between $\pm p_0 \tan\theta'$; also $p_0 \tan\theta' > p_0 \sin\theta' = p_0 (\lambda/\tau)$.

Now we don't know and cannot follow in detail (for the same reasons that prevented our closer examination of the formation of the two-slit pattern), how any single electron is deflected in passing the slit. The foregoing analysis shows that after passage of an electron we can only assert that the y component of momentum is found somewhere within the range $\pm p_0\lambda/\tau$, which, since $\lambda = h/p_0$, is $\pm h/\tau$. Thus we cannot predict with precision what the value of the measured y component of an electron which gets through the slit will be. We shall set the certainty[14] or dispersion of the y component of the momentum equal to Δp_y; its value is roughly $\gtrsim h/\tau$. Note that τ also measures an uncertainty. We cannot predict where a single electron will come through the slit but obviously it must be found somewhere within the width τ of the slit. τ then roughly determines an uncertaintu in the y component of position of an electron in passage through the slit. Setting $\tau \cong \Delta y$ we have $\Delta p_y \approx h/\Delta y$, or

$$\Delta y \Delta p_y \approx h. \tag{4.1}$$

The result may be summarized as follows: If an electron of known momentum passes a slit of width Δy, we can only predict the y component of the momentum after passage to within an uncertainty Δp_y, where Δy and Δp_y are related by (4.1). For an electron with an energy of the order of 100 eV ($2\lambda \approx 1\text{Å}$), the uncertainty Δp_y introduced in passage through a slit of macroscopic size (say $\Delta y \approx 1$ mm) is negligible by comparison with the momentum $p_0 = \sqrt{2mE}$; we have $\Delta p_y/p_0 \cong h/p_0\Delta y = \lambda/\Delta y \approx 10^{-8}/10^{-1} = 10^{-7}$. But if Δy is of the order of angstroms the uncertainty Δp_y is of the same order as p_0 itself.

Equation (4.1) suggests that if we arrange to determine a component of the position of an electron we can do so only at the expense of our prior knowledge of the corresponding component of the momentum. If this implication of our work is generally true it is of enormous significance. To make a prediction in classical mechanics we must be given "initial conditions"; for a single particle these conditions are the position and momentum. If we cannot know both the position and momentum of the particle our mechanics loses its power to predict. Now Eq. (4.1) follows essentially from the de Broglie relation, $\lambda = h/p$, and the meaning of λ in relation to diffraction effects. Do the de Broglie relations imply basic limitations on our capacity to determine the position and momentum of an electron simultaneously? We turn now to this question.

---

[14] The "uncertainty" is defined as the root mean square of the deviation from the mean value. If the distribution of the y component of the momentum for an ensemble of electrons diffracted by the slit is $w(p_y)$, i.e., if $w(p_y)dp_y$ is the probability of finding the component p_y in the interval dp_y, then the uncertainty is $\Delta p_y = [\int w(p_y)(p_y - \bar{p}_y)^2 dp_y]^{\frac{1}{2}}$ where $\bar{p}_y$ is the mean value of p_y in the distribution. In the case we are considering, $\bar{p}_y = 0$.

5 HEISENBERG'S PRINCIPLE OF INDETERMINACY

> "O, swear not by the moon, the inconstant moon."
> —William Shakespeare

We want to measure both the position and momentum of an electron. Let's try to make the problem as simple as we can. Suppose that the momentum has been measured and that after the measurement the electron is "free" so that the electron momentum does not change with time. If the measured component along the x axis has the value p_x, then a subsequent measurement of the x component of momentum is certain to yield the value p_x. All that now remains is to measure the position without, in the process, changing the momentum, or if the position measurement does alter the momentum, to measure the position in such a way that after the measurement both the position and the momentum are known. To make the position measurement we use a microscope.

The stage of the microscope is illuminated with radiation of some definite wavelength, λ, directed along axis (Fig. 5.1). A fluorescent screen that scintillates when a photon of wavelength λ falls upon it is placed in the microscope so that an image of an object illuminated by the radiation is formed on the screen. We observe the flourescent screen through the eyepiece. If we are lucky we discover, after some period of watching, a scintillation at some point P, on the screen. This effect results from the scattering of a photon in the incident radiation by the electron into the microscope. From the observed location of the scintillation we must determine the location of the electron at the time the radiation was scattered. Of course the scattering event will change the previously measured electron momentum so that we must also determine, if possible, the new value of the momentum of the electron after the measurement.

It is well known that a point source of radiation does not produce a point image in a microscope [10, chap. 4]. Diffraction effects lead to an image that is spread over a small circular region. The radius of this circular image depends on the diameter, d, of the objective lens (see Fig. 5.2) and the wavelength of the radiation

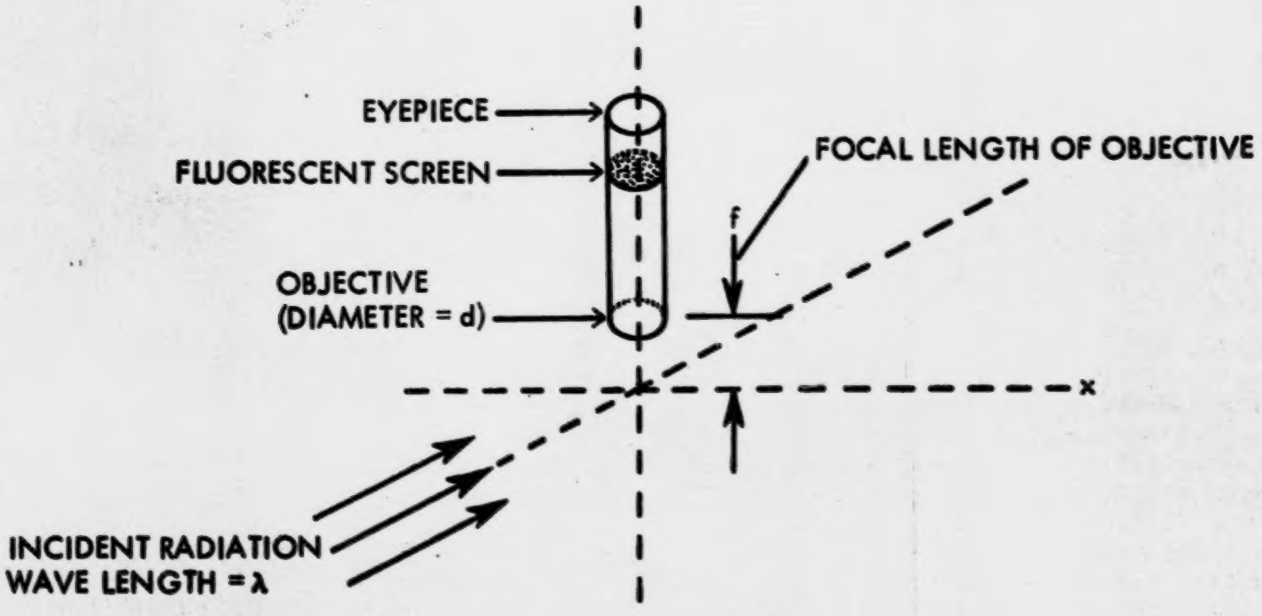

Fig. 5.1 Position measurement with a microscope.

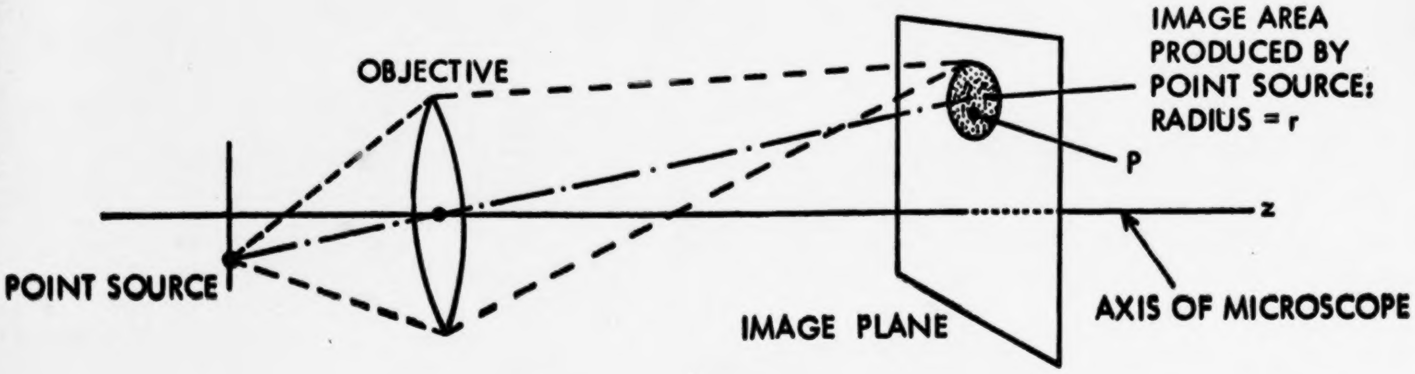

Fig. 5.2 Image of a point source in a microscope.

from the point source. The first minimum of the diffraction pattern formed on the image plane for a point source of radiation occurs at a distance r from the center of the pattern. Roughly speaking, this minimum is formed where the paths of the radiation that pass through opposite ends of a diameter of the objective (see the dotted paths in Fig. 5.2) differ by a wavelength. It is not difficult to show[10, chap. 4] that the angle subtended by the image area of radius r at the objective lens is given approximately by

$$\delta = \lambda/d, \qquad (5.1)$$

where d is the diameter of the objective lens. It must be realized that the areal image of a point source is produced by a large number of photons striking the image plane at well-defined points (or, more exactly, regions small compared with the area of the image). In the observation of a single scintillation, a single photon is detected at P (say). Photons scattered from a range of different locations may give rise to scintillations at P. Thus the scintillation at P might have arisen from a photon emitted from sources at S_1 or S_2, or from any point between S_1 and S_2 (see Fig. 5.3). The observation of a scintillation at P implies that the photon was scattered from some point along the axis within the range Δx, (Fig. 5.3). This uncertainty in the location of the point at which the electron scattered the photon is approximately δf, where f is the focal length of the objective, and δ is given by (1); consequently

$$\Delta x \approx \lambda(f/d). \qquad (5.2)$$

Consider now the change in the electron momentum produced when the photon that reaches P is scattered by the electron. This process (Comton effect) may be described by assuming that the incident photon is a particle with momentum h/λ and energy $E = h\nu$ ($\nu = c/\lambda$) and that momentum and energy is conserved in the collision. To get into the microscope the photon must be scattered into a cone of angle θ (Fig. 5.4). Thus the components of the momentum of the scattered photon along the x axis lie between $\pm (h/\lambda) \sin \theta$. (We neglect the change of λ in the collision.) Also $\sin \theta \approx \tan \theta = d/2f$. Since the

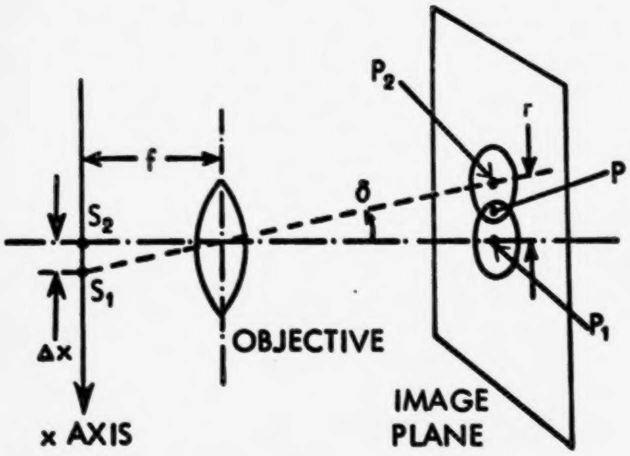

Fig. 5.3 Determination of position uncertainty in microscope measurement.

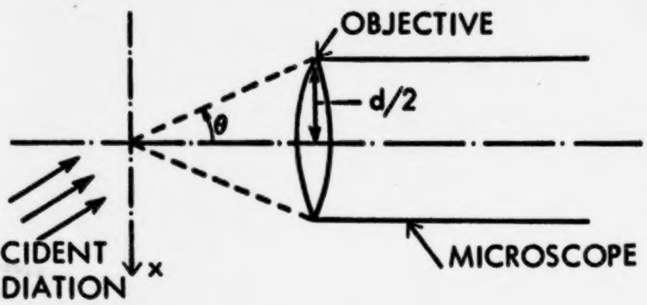

Fig. 5.4 Momentum uncertainty after position measurement.

total momentum of electron and photon is conserved we may conclude that the momentum after the observation lies between $p_0 + h/\lambda\ (d/2f)$ and $p_0 - h/\lambda\ (d/2f)$ where p_0 is the measured momentum before the scattering. The uncertainty in the x component of the momentum of the electron after collision is, therefore,

$$\Delta p_x \approx \frac{h}{\lambda}\frac{d}{f}. \qquad (5.3)$$

For the product of the uncertainties we find

$$\Delta x\, \Delta p_x \approx (\lambda f/d)\ (h/\lambda)\ (d/f) = h, \qquad (5.4)$$

which is just what was found from the analysis of the single-slit diffraction (Eq. (4.1))! The uncertainties in the position and associated momentum component of the electron as measured by the microscope are inversely related. If we arrange to make Δx small (by making λ small or θ large), Δp_x gets large; similarly if Δp_x is made small (by making λ large θ small), Δx becomes correspondingly large. If the position of an electron with an energy of the order of tens or hundreds of electron volts is to be determined within an uncertainty of macroscopic size ($\Delta x \sim 1$ mm $= 10^7$ Å) the uncertainty in p specified by (4.1) is negligible: Using Eqs. (4.1), (2.6), (2.7),

$$\Delta p_x/p_x = h/p_x\Delta x = \lambda/\Delta x \cong 12.3/E^{\frac{1}{2}}\Delta x$$
$$\sim 12.3/10\cdot 10^7 \sim 10^{-7}.$$

If, however, we wish to determine the position of an electron within an uncertainty as large as an atom ($\Delta x \sim 1$ Å) the uncertainty in p_x is of the order of p_x.

The relation (4.1), first derived by Heisenberg[20,21] by an analysis of the observation of position with a microscope and other modes of position, momentum measurement, is known as the Heisenberg uncertainty relation. Heisenberg's discussion of the uncertainty relation contributed greatly to an understanding of the physical meaning of the formal structure of quantum mechanics. The mathematical formalism of quantum mechanics was discovered in 1925 by Heisenberg, Schrödinger, and Dirac. It took about two years after this discovery to appreciate the full physical significance of the new formalism.[15]

Of course the failure of the microscope method does not imply by itself that a precise simultaneous measurement of position and momentum is impossible. Perhaps the desired measurements can be made by other methods. However, the analyses of all proposed devices for making the measurements - a number of most ingenious devices have been investigated - lead uniformly to Heisenberg's relation (4.1). These investigations suggest strongly that the limitation specified by the uncertainty relation expresses a general law applicable not only to electrons and photons but to all microphysical entities.

In the argument leading to Eq. (5.4) a wave theory was used to obtain the position uncertainty,[16] Δx, while to obtain the momentum uncertainty, Δp_x, a particle theory of the collision of a photon with the electron was employed. This admixture of wave and particle considerations occurs in the analysis of every method designed to measure position and momentum. Naturally, a paper and pencil analysis of a measurement presupposes some theoretical foundation. Exactly what are the theoretical assumptions employed? They are simply the de Broglie relations with the following interpretation: in processes of energy and momentum transfer, electrons and photons (as well as other microphysical entities) are to be treated as particles, while in processes of transmission between spatially separate interactions they are to be treated as waves. All relevant observations are consistent with this interpretation of the de Broglie relations.

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[15] It is an interesting feature of the historical development of quantum mechanics that the mathematical structure of the theory was invented before its physical interpretation was completely understood. One would expect a new theory to grow out of new physical insights into the nature of phenomena, with the mathematical structure built afterwards to give precise quantitative expression to these insights. Some of the reasons for the reversed order in the case of the quantum mechanics are considered in sections 6, 7, 8.

[16] The diffraction in the microscope was analyzed by a wave theory. Remember, however, that this theory does not describe all the features of the diffraction of photons.

5.1 SUPPLEMENT TO SECTION 4.

In our investigation of the two-slit diffraction experiment we postponed consideration of the effects attendant on observations designed to determine whether the electron goes through slit a or slit b, or possibly, in some manner,, through both (see Fig. 4.7). Suppose we arrange devices capable of detecting a passing electron in front of the slits (see Fig. 5.5). To be specific, let's suppose that we have a source S emitting radiation which illuminates both slits. We have also a pair of microscopes M_a, M_b which are focused on the region in front of the slits. Of course we arrange that if the electron beam is off, no photons can be scattered into the microscopes.

To be able to distinguish between the passage of an electron through slit a or slit b, we must be sure that the detection of a photon in microscope M_a cannot be interpreted as having entered M_a after being scattered by an electron in the neighborhood of slit b. The microscopes must be capable of locating the electron within an uncertainty Δy smaller than the separation, d, between the slits. Such a measurement gives rise to an uncertain change in the momentum of the electron; the uncertainty in the y component of the momentum introduced by the measurement is

$$\Delta p_y \cong h/\Delta y \gtrsim h/d.$$

This momentum uncertainty is equivalent to an uncertainty, $\Delta\theta$, in the direction taken by the electron after passage through the slit. This angular uncertainty, $\Delta\theta$, is roughly $\sim \Delta p_y/p$ where p may be taken as the magnitude of the incident momentum. Thus, making use of the de Broglie relation, $\lambda = h/p$,

$$\Delta\theta \sim \Delta p_y/p \sim h/p\Delta y \gtrsim h/pd = \lambda/d.$$

But the angular separation at the

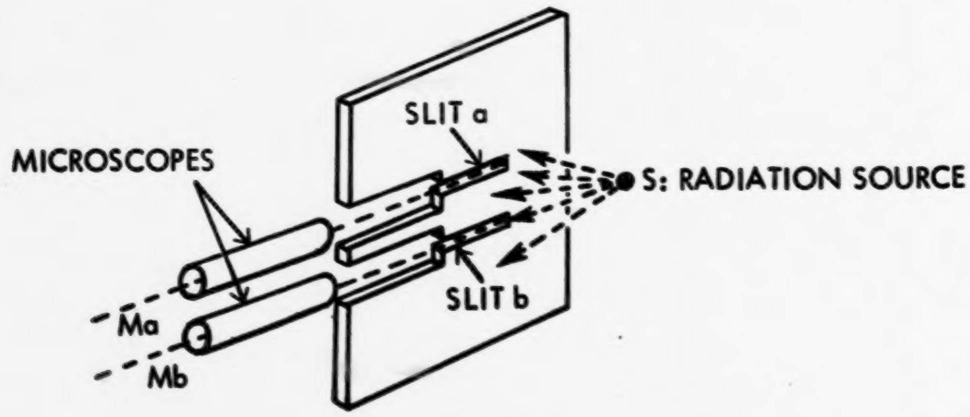

Fig. 5.5 Observation to determine through which slit an electron passes.

slits of successive maxima in the interference pattern when the detectors are absent (see Fig. 4.11) is also λ/d. We have therefore $\Delta\theta \simeq \theta$, where θ is the angular separation (measured from the slits) of successive maxima in the normal two-slit pattern. By the position measurement we introduce a new directional uncertainty (superimposed on the directions taken by the electrons after getting through the slits), of such size as to smear out the characteristic structure of the two-slit pattern.

5.2 CONTINUATION: HEISENBERG'S PRINCIPLE

Assume now that the Heisenberg relation (4.1) expresses a fact of nature, i.e., that the laws of physics make it impossible to know both the momentum and position of an electron with uncertainties smaller than those permitted by Eq. (4.1). What consequences follow? Can the uncertainty relation help us to see how a rational theory of microphysical processes might be formulated?

One consequence is immediate, of obvious importance, and is independent of further theoretical assumptions. In principle, *either* the position *or* the momentum of an electron can be measured with arbitrary accuracy; the accuracy within which both properties can be known simultaneously, however, is limited by Eq. (4.1). Suppose that the momentum of an electron has been measured precisely; according to the Heisenberg relation, the position at which the electron will be found on measurement cannot be predicted. However, in an exact observation of position the electron is found at some definite location (after which the result of a momentum measurement cannot be predicted). The point of all this is that the Heisenberg relation, or *Heisenberg principle*, implies that it is not possible to arrange initial conditions so as to be able to predict the results of all possible observations that might be made on an electron. If position (or momentum) is known the result of an exact momentum (or position) measurement is unpredictable. If position and momentum are known within the uncertainties Δx, Δp_x, the result of either an *exact* position measurement or an *exact* momentum measurement cannot be predicted although, of course, a precise measurement of position will yield a result somewhere in the range Δx, *or*, if momentum is measured precisely, its value will be found in the range Δp_x.

This unpredictability of the properties of an electron may remind us of the unpredictable features noted in the study of diffraction phenomena. Equipment could not be arranged so as to be sure that an electron that gets through the slits will arrive at some particular point on the detecting screen. (Note that an observation of a scintillation on the screen is a position measurement for the electron.) The Heisenberg principle suggests that the indeterminate behavior of electrons encountered in the diffraction study is by no means peculiar to the phenomenon of diffraction. *Indeterminate behavior*, it would seem, *must be expected throughout the realm of microphysics*. But how can one possibly have a science for indeterminate, i.e., unpredictable behavior?

The diffraction studies of Section 4 suggest an answer. The positions at which individual electrons appear on the detecting screen in the diffraction process cannot be predicted, or, put in another way, the diffraction apparatus cannot be arranged so that every electron in the incident beam will arrive at the same point on the detecting screen. However, and this is the crucial consideration, the distribution of position at the detector, i.e., *the diffraction pattern*, can be predicted. If twenty physicists perform the diffraction experiment at different places, different times, they all find the same pattern. The nature of the electron source, the manner of detection, the methods used for producing the incident beam have no influence on the pattern. Whenever and however an incident beam with a given well-defined momentum is incident on a given crystal scatterer, the same distribution in position at the detector is found; the incident momentum uniquely determines the final distribution. It is clearly possible to predict not the locations of single electrons but the distribution in position of a large collection or ensemble of electrons.

In effect a theory of microphysical processes must have a statistical character. In general, only statistical properties, i.e., ensemble properties, rather than the properties of single entities are determinate in the realm of microphysics. Often a somewhat loose use of language obscures this fact. We have already explained what is meant when we speak of the wave property of electrons as exhibited in the diffraction experiments. It is not a single electron with definite momentum that is similar to a wave but, rather, a large ensemble of electrons all with definite momentum. An additional example of the determinate properties of ensembles of microphysical entities will be helpful. A free neutron undergoes spontaneous transformation into a proton with the emission of an electron and a neutrino. It is often said that the "half-life" of a neutron is 12 minutes. This is a statement not about individual neutrons but about ensembles of neutrons. If, initially, we have a set of neutrons (at rest), some transform before one minute has passed while others have stubbornly refused to change into protons even after the lapse of an hour. We cannot predict, precisely, when any individual neutron will change to a proton, but after 12 minutes about half of the original collection of neutrons will be changed into protons. Regardless of the prior history of the neutrons collected in an ensemble, half the ensemble will be transformed into protons at the end of 12 minutes. The term "half-life," obviously, is an ensemble property.

Let's summarize briefly. The Heisenberg principle implies that there is an irreducible indeterminateness in the behavior of microphysical systems. Experimental conditions cannot be arranged so as to be certain of the outcome of all observations. However, the behavior of suitably prepared ensembles is found to be regular and lawful.

6 INTERPRETATIONS OF THE HEISENBERG PRINCIPLE

> "All things counter, original, spare, strange,
> Whatever is fickle, freckled (who knows how?)
> With swift, slow; sweet, sour; adazzale, dim;"
> —Gerard Manly Hopkins

The general significance of the Heisenberg principle may be variously interpreted. Two classical attitudes toward the principle are sketched in subsections 6.1 and 6.2. An interpretation that breaks sharply with classical conceptions - the interpretation that leads to quantum mechanics - is introduced in subsection 6.3.

6.1 CLASSICAL STATISTICS.

Although the position and momentum of an electron cannot be *measured* simultaneously it still may be presumed that an electron has, at each instant, both a well-defined position and a well-defined momentum. In other words, the uncertainty principle in itself does not prevent us from considering the electron as a Newtonian particle. Because of the peculiar consequences of the wave particle dualism, we simply cannot *know* both the position and the momentum of an electron at an instant. But, since position and momentum are the initial conditions required to predict the path of an electron, accurate prediction is impossible and the motion of an electron is uncertain. We may, however, try to make statistical predictions by the methods of statistical mechanics. When we deal with a sample of a gas, we do not know, for practical reasons, the positions and momenta of the molecules that constitute the gas. Nevertheless relations between such statistical properties as the pressure and the temperature of the gas (for example) can be derived.

When an object subject to some set of conditions is studied by the methods of statistical mechanics - whether it's a sample of a gas or a single electron - a large ensemble of objects of the same kind (all subject to the same conditions), rather than a single object, is considered. The ensemble may be a theoretical entity (as in the case of a gas sample, since we don't collect for study 10^9 or so samples of the gas), or a natural entity, as is often the case in microphysics where observations on a single system (atom, electron) are impractical, while observations on large collections of these systems are relatively simple. In any case we attempt to calculate, or observe (if a physical ensemble is available) the *distribution* of the properties of interest over the whole ensemble rather than the properties of a single object.

We review the statistical method for treating a single electron (now considered as a Newtonian particle), moving in one dimension. It will be useful for this purpose to introduce a "phase space" diagram (Fig. 6.1). A point in this diagram (a "state point") represents a single electron with position as given on the abscissa, and momentum as given on the ordinate. Suppose our electron to be subject to an external force of some kind. The motion of the electron is completely determined once its position and momentum at some time - say t = 0 - are given. Both position and momentum change, in general, under the influence of the external force and hence the state point in the phase space diagram traces out a definite path.

When we have imperfect knowledge about the initial state of the electron we introduce an ensemble of electrons, each with some definite position and momentum, so as to represent

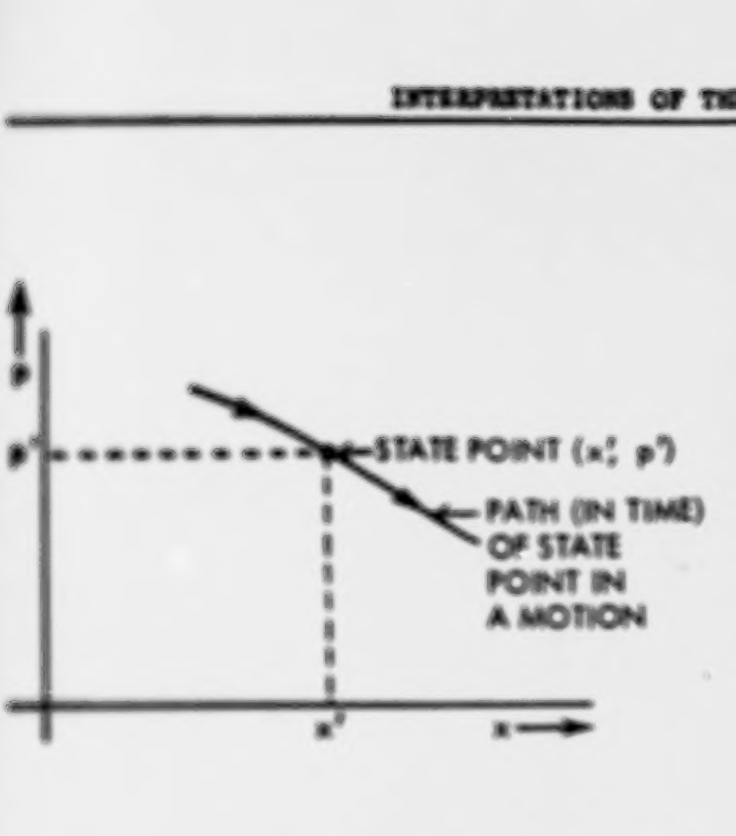

Fig. 6.1 Phase space diagram for motion of a point particle in one dimension.

the range of possible initial conditions that are consistent with the available information; i.e., each electron in the ensemble chosen has a chance of being the electron we wish to study. An ensemble is represented in the phase-space diagram by a large number of points each of which is a state point for a single electron. Figure 6.2 shows a phase-space representation of an ensemble that could be used to study the motion of an electron (in the statistical sense) if all that is known is that the uncertainty in x is Δx about the neighborhood of x_0, the uncertainty in p is Δp about the momentum p_0, and the x and p distributions are uncorrelated. (Absence of correlation means that the distribution of the points along the p axis is the same for every chosen value of x within the range Δx.)

From the assumed initial ensemble and the laws of motion, the position of all state points at any later time, and the distribution in x, or in p, or in any other dynamical variable of interest can be calculated. Thus the initial ($t = 0$) characteristics of the ensemble determine the average value of the position in the ensemble at time t, the uncertainty in position at t, etc.

The Heisenberg principle asserts that when measurements are made as accurately as possible, there remain position and momentum uncertainties Δx, Δp such that $\Delta x \Delta p \sim h$. Therefore to make a statistical analysis of the motions of an electron, it must be replaced by an ensemble with a phase space representation like that of Fig. 6.2. (We use an uncorrelated distribution since observations can tell us nothing about possible correlations). Note that the Heisenberg principle says that no matter how the position and momentum of an electron are measured, the ensemble of phase-space points required for statistical treatment must cover an area in phase space roughly of magnitude $h (\cong \Delta x \Delta p)$.

It seems highly improbable that the foregoing classical statistical theory could solve the conceptual problems raised by microphysical observa-

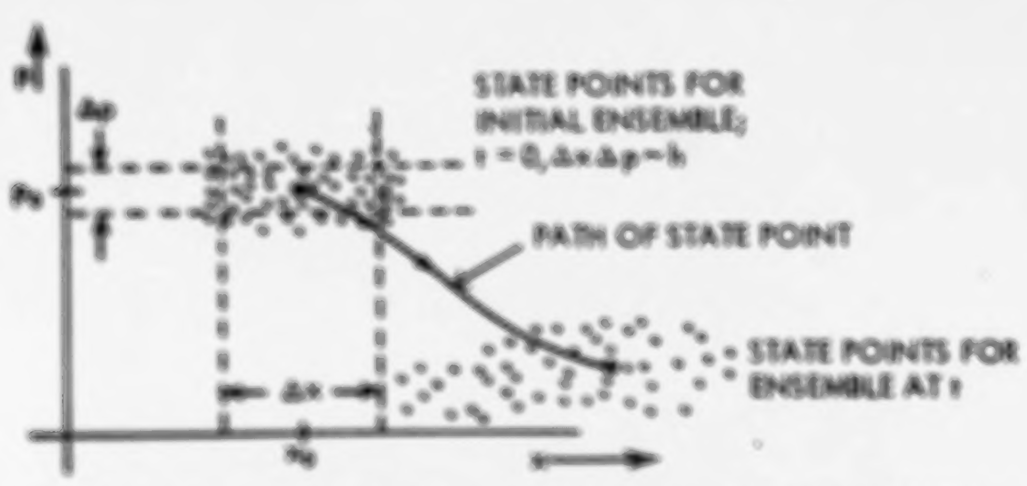
p
Δp
p₀
STATE POINTS FOR INITIAL ENSEMBLE;
t = 0, ΔxΔp ~ h
PATH OF STATE POINT
STATE POINTS FOR ENSEMBLE AT t
Δx
x₀
x

Fig. 6.2 Representation of an ensemble of electrons in phase space.

tions. The phenomena of quantization would still require the addition of restrictive rules of some kind (as in the theory of Bohr); the introduction of such rules would lead to the same objections as were generated by the Bohr theory (see Section 1). The difficulties are more apparent if we consider the wavelike properties of electrons. How, for instance, could a statistical theory handle the two-slit diffraction experiment? The theory considers electrons to be classical particles; each electron would be pictured as going through one *or* the other of the two slits. Thus the theory could only lead to the false prediction that the two-slit pattern is a simple superposition of two one-slit patterns.

6.2 HIDDEN VARIABLES

The Heisenberg principle implies that the behavior of electrons is indeterminate. What is the cause of this indeterminacy? Are we to understand that on the microphysical level there is an essential play of chance, or, to paraphrase Einstein, that "God plays dice with elementary phenomena"? "Chance" is merely a word we use when we try to hide our ignorance. Surely (says the classical physicist) the position at which each electron hits the screen in a diffraction experiment is uniquely determined by *some* specific set of conditions, even if we have not yet discovered exactly what these conditions are.

Macrophysical systems may seem to have an indeterminate behavior if we do not take into consideration all the details of their structure. Suppose, for example, that we have a set of boxes all with the same dimensions and all of the same mass containing differently oriented gyroscopes with different angular momenta. Under external torques of the same magnitude and direction (relative to the box geometry) different boxes will exhibit different responses. An observer who is able to examine only the outside features of this collection of boxes might claim that they are identical but that their behavior is indeterminate. A physicist who made such an interpretation without even thinking that the boxes might be systems with different internal properties would be a simpleton. We who know about the internal gyros have no difficulty in explaining the variation in the behavior of the different boxes. Obviously if we don't know and don't control all the variables on which the behavior of a set of systems depends, the same external influences may lead to different consequences simply because the different systems have different internal properties.

Might this not be the origin of the indeterminacy implied by Heisenberg's principle? We don't really know in any direct sense what an electron is. When a photon registers in the microscope we speak of a "position" measurement. From the "position" measurement alone it is not possible to predict the result of a subsequent "momentum" observation. But might there not be as yet unknown measureable properties, i.e., *hidden variables* (corresponding to the gyros in the example above), which, if measured together with "position," would enable us to predict the result of a subsequent "momentum" measurement?

It is impossible to prove or disprove the existence of the hidden variables needed to make the behavior of microphysical entities determinate.[17] As yet, however, no one has turned up any trace of such hidden properties. It is only the brute fact of the appearance of indeterminacy in the behavior of microphysical systems that suggests the existence of hidden variables. The proof of a pudding, it is said, is in the eating. Forty years ago (1925) a highly successful nondeterministic theory (quantum mechanics) of microphysical systems was formulated. In all the work which has gone on since 1925 no evidence for the existence of hidden variables that might serve to reestablish determinism has been found.[18]

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[17] An assertion may be proved or disproved only relative to some accepted theoretical structure. It has been shown (the proof is given by von Neumann[22]) that if certain general assumptions employed in quantum mechanics are true, "hidden variables" capable of making the theory determinate cannot exist. However, there is no law against the belief that this simply reflects a fault in quantum mechanics. Some of the foremost contributors to the theory of quantum mechanics, including Einstein, Schrödinger and de Broglie, never fully accepted its indeterministic character.

6.3 A NONCLASSICAL INTERPRETATION OF THE HEISENBERG PRINCIPLE.

The Heisenberg principle states that the exact simultaneous values of the position and momentum of an electron cannot be observed. Recall also that the motion of electrons in diffraction processes is not observable; it could not even be determined through which of two slits each electron passes, without completely changing the resulting diffraction pattern. Now the concepts of simultaneous position and momentum, and of a path through one slit or another derive from the conception of an electron as a classical (Newtonian) particle. Are there other features of the mental images we form of microphysical processes that are unobservable? From the Heisenberg principle it follows that a large number of the concepts of classical physics have no observational counterpart in the realm of microphysics.

Suppose we wished to observe the path or orbit of the electron in a hydrogen atom. To measure the path we must determine the positions of the electron at a number of successive instants. Since the atom has a diameter of about one angstrom, any meaningful position measurement must be made with an uncertainty, Δx of no more than about .1Å. By the Heisenberg relation such a position measurement introduces an uncertain change in the electron's momentum of about $h/\Delta x$, where $\Delta x \sim 10^{-9}$ cm. A momentum change of this magnitude corresponds to a change in energy which is about ten times the ionization energy of hydrogen. Thus the first measurement of the sequence required to determine the electron's orbit is almost certain to ionize the atom. In principle we may determine the position of the electron (with arbitrary accuracy), in the atom at some time, but we cannot observe the electron path.

A further deduction may be made from this example. In the analysis of the impossibility of observing the orbit we noted that the position measurement resulted in the possibility of a large and uncertain transfer of energy to the atom (corresponding to an uncertain alteration in momentum). This suggests that the exact energy of the atom and the position of its electron cannot be known simultaneously. Suppose the energy has been measured. After the measurement the energy is constant; hence we should know both energy and momentum if the position can be measured without change, or with a determinate change, of the energy. The Heisenberg principle implies that these requirements cannot be met. The measurement of the position will alter the energy of the system in a manner that we cannot completely control. By analogous considerations it could be shown that the energy and momentum of an electron in an atom cannot be known simultaneously. Our inability to measure pairs of variables simultaneously is by no means limited to the position, momentum pair; indeed our findings suggest that simultaneous measurability of quantities is the exception rather than the rule in microphysics.

In classical mechanics the concepts of velocity and acceleration play important roles. Force is related directly to acceleration, and the force on an object is measured by observing, in one aspect or another, the acceleration the force induces. But, at the level of atomic dimensions, velocities, accelerations and consequently forces are unobservable.[19]

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[18] Much more remains to be said on this issue. Further discussion is postponed to Chapter 2 where the question of hidden variables will be examined in relation to a specific example.

We have considered how the Heisenberg principle (deduced from the de Broglie relations) affects the observability or measurability of features of classical mechanical systems. Classical electromagnetic concepts may be investigated in the same spirit. We shall not attempt an analysis here but simply quote results[21]. Electric and magnetic fields cannot be measured simultaneously with accuracy. To predict, by classical theory, the development in time of an electromagnetic field it is necessary to know both fields at the same time. Our inability to measure the fields simultaneously thus undercuts the possibility of prediction in electromagnetism in the same way that our inability to measure position and momentum simultaneously prevents prediction for mechanical systems.

Consider a somewhat different consequence of the Heisenberg principle. When we think the word "electron" we are likely to generate a mental picture of a tiny object which has a definite location in space and is either at rest or moving in some direction with a definite speed relative to a coordinate frame. Is the physical electron an entity that corresponds to this picture? Maybe it is. But we can't establish the truth of the picture by observation. Our mental model implies that the electron is an entity capable of having both a precise position and a precise momentum. But since position and momentum cannot be measured simultaneously and exactly, we cannot prove that the electron can have both a precise position and a precise momentum.[20]

It is possible to go considerably further in the direction taken by the foregoing paragraphs. However, a sufficient number of instances have been cited to indicate that only a remarkably small subset of classically meaningful properties are observable in microphysical phenomena. What are we to make of all these examples of our incapacity to observe and measure? It is depressing to think of so many failures. But before we go off to cry in our beer over the decay of physics let's remember that the failure to observe an expected phenomenon does not necessarily reflect an incapacity of some kind; it may be that the *phenomenon just wasn't there to be observed*!

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[19] If the position components at t_1, t_2 are x_1, x_2 then the average velocity in the interval $t_2 - t_1$ is $x_2 - x_1/t_2 - t_1$. In order that this average velocity be a reasonable approximation to the electron velocity in the interval $t_2 - t_1$, the distance $x_2 - x_1$ must be small compared with the distance over which the velocity changes markedly; thus $x_2 - x_1$ must be considerably smaller than an atomic diameter, since in moving across an atomic diameter the velocity must change sign. Take $x_2 - x_1 \sim .1$Å. To obtain a measure of this difference with about ten percent accuracy the positions x_1, x_2 must be known with uncertainties no larger than .01Å. Such accurate position measurements lead to large and uncertain momentum and energy changes. For a position accuracy of .1Å we calculated that the associated energy uncertainty is about ten times the ionization energy; for an accuracy in position of .01Å the momentum uncertainty is increased by a factor of ten, and, since the energy depends on the square of the momentum the energy alteration can be as large as (100) (10) = 1000 times the energy required for ionization. Clearly the atom will be ionized in the first position measurement and our effort to obtain the velocity of the electron in its orbit fails completely. Since acceleration is the rate of change of velocity and velocity is not measureable, neither is acceleration. Moreover, force, which is determined by the acceleration it produces, cannot be measured.

The preceding remarks refer to electrons within atoms. If, as in many electron beam experiments (see the next to the last paragraph of this section), it is sufficient to measure position to an accuracy of the order of a millimeter (rather than 10^{-8} mm as is required to obtain the path within an atom), the concepts of path, velocity, acceleration, force can be given approximate meanings. Anyone who has seen tracks of electrons in cloud chambers or photographic emulsions has had visible proof that the concept of electron path is sometimes meaningful.

[20] When the restrictions of the Heisenberg principle are not significant, position and momentum can be measured with enough accuracy as to make a particle model of the electron a good approximation (see Chapter 2).

Let's return then to the usual optimism of physicists and deny failure. Instead of looking upon the Heisenberg principle as an indicator of incapacity we shall boldly assume that an electron is an entity that simply *does not have an exact simultaneous position and momentum*. (To avoid its frequent repetition, the awkward phrase, "exact simultaneous position and momentum," will be designated by "X".) On our new assumption the words, "an electron with the property X," is as meaningless as the phrase "a square of radius r." This assumption seems harmless enough - after all, the analysis of diffraction phenomena in Section 4 indicated clearly that an electron isn't a simple Newtonian particle - but it contains packages of dynamite. Consider an electron that has been prepared with a precise momentum; our assumption forces the conclusion that *this electron does not have a position*. For if the electron with definite momentum is located somewhere, then it has both position and momentum simultaneously (even though the position is unknown), i.e., the electron has the property X, and this, by the assumption introduced above, is impossible. An argument of the same kind would show that an electron with a definite position does not *have* a momentum property. Clearly, if X is not an electron property, electrons do not have positions or momenta under all circumstances.

It is not difficult to invent situations with features analogous to those of the preceding paragraph. For example a plane string figure cannot have simultaneously both a precise "radius" and a precise "side length." If the string figure is a circle it has a radius but it does not have a property of side length; if, however, the figure is a square than it has an exact "side length" but it does not have a radius. For a second example consider certain motions of an infinitely long stretched string. If the motion is characterized by an exact wavelength, no meaning is associated with the idea of a sharp location of the wave; when the wavelength is exact the wave is infinite in extent (Fig. 6.3a). When, however, a sharply defined pulse (Fig. 6.3b) travels down the wire, the position of the disturbance (at some instant) has a fairly well-defined meaning, but the concept of wavelength loses significance.

However, the problem posed by the

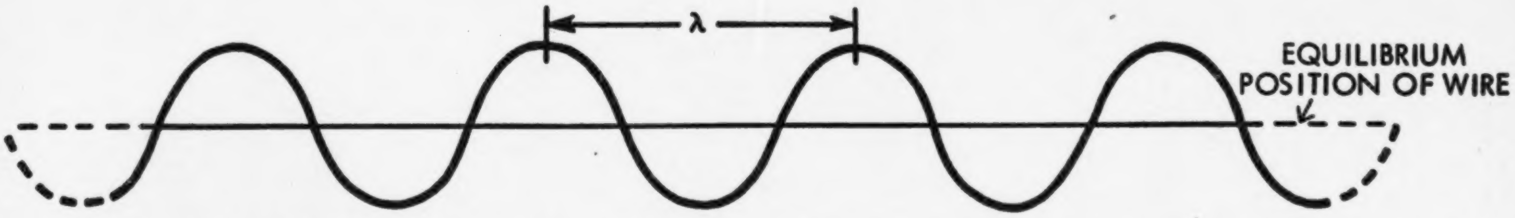

Fig. 6.3a The deflection of a wire at some instant during a motion. At this instant, the motion has a wave length but not a location property.

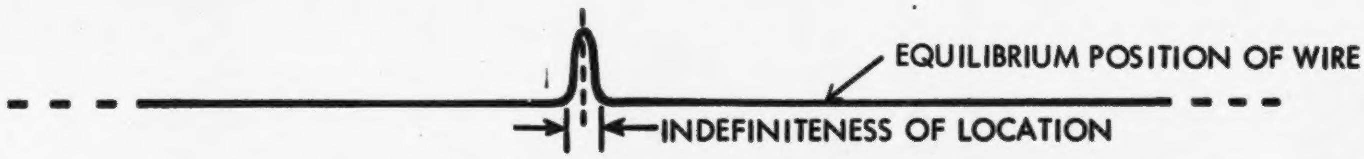

Fig. 6.3b The deflection of the same wire as that of Fig. 6.3a at some instant in a different motion. The disturbance has a well defined location but has no wavelength property.

assumption that X does not exist for electrons has features the examples above fail to encompass. There is no reason to doubt that, whatever the circumstances, a position (or momentum) measurement can always be made successfully on an electron. In consequence we are led to what seems an absurd conclusion: although an electron with precise momentum cannot be said to have a position, nevertheless an exact position measurement yields some result (the words "position" and "momentum" may be interchanged). For classical physics such a conclusion is nonsensical; it simply isn't possible to measure a property that an object does not have. For example no one would dream of measuring the wavelength of a classical particle; the measurements that can be made on such a particle could never lead to a wavelength. Similarly it is not possible to find on observation a wave with a precise wavelength at a sharply defined location.[21]

The ideas developed above, if disturbing, are nevertheless not without attractive features. In the two-slit diffraction experiment (Section 4) both slits appear to have an influence on the behavior of the electrons that get through; the two-slit pattern is quite different from the sum of two single-slit patterns. If the electron always has a sharply defined (even if unknown) location it seems to be impossible to understand how an electron can "know" whether the slit through which it does not pass is open or closed. Recall that the incident electrons in the diffraction experiments must have well-defined momentum. By the assumption considered in the previous paragraphs the electrons in the incident beam do not have a sharp location. It is therefore at least conceivable that both slits play a role in the transmission process. Of course these thoughts don't constitute a theory of the diffraction experiments; they simply indicate that certain of the paradoxes considered in Section 4 can be avoided if position is not always a meaningful property of electrons.

Is it possible, however, to conceive that an electron does not always have a definite location even though a measurement of position will always discover it at some place? The training we have all had from birth in macrophysics makes this idea very difficult to accept; it is in fundamental conflict with deeply held presuppositions about the meanings of the concepts of "measurement" and "property." Unless these presuppositions can be shown to be false, the assumption that electrons cannot have a precise simultaneous position and momentum must be abandoned.

The classical assumption relating to the concepts of measurement and property are criticized in the next section where it is shown that the ideas developed in the preceding paragraphs are neither internally inconsistent nor controvertible by observation. For our present introductory purposes the following brief but suggestive remarks must suffice. In an accurate position measurement strong interactions occur between the electron and the apparatus employed to effect the measurement. In the microscope method (Section 5), for example, an energetic photon must be scattered off the electron; the more accurate the measurement the more energetic the photon must be. Just how the rough treatment suffered by an electron in the course of measurement modifies its characteristics cannot be known. All that we have available are observable manifestations - scintillations, counts - from which properties are inferred. After a suitable position measurement on an electron with a definite momentum, the result of an immediate repetition of a position measurement can be predicted; the electron is said, therefore, to have a position property (see Section 7). (After the measurement the electron no longer has a momentum property.) But (by the very construction of the statement) *it is impossible to establish by observation* that just prior to the position measurement the electron had the location at which it was subsequently discovered.[22]

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[21] On the plane string figure of radius r it is senseless to measure "side length."

It is useful to recall that the central object of theory is to provide a set of general rules from which the observable relations among phenomena may be deduced. No theory legitimately can be asked to do more. Certainly theories need not be required to make statements about imagined but unobservable features of phenomena. Anyone who asserted that electrons are red but that the property of redness is unobservable, and then went on to wonder how the redness of electrons influences other unobservable properties of microphysical phenomena would be considered slightly mad. There may, however, be good reasons for the introduction of unobservable properties in theoretical considerations. If, without affecting its predictive power, we can make a theory simpler, or applicable to a wider range of observable phenomena, or more beautiful, by introducing unobservable features, why not do so? No one is bothered, for example, by the unobservability of the "inside" of a completely closed box or the widely used but unobservable concept of continuity. Equally, when theory fails, as classical theory fails in the realm of microphysics, its imagined but unobservable aspects may be cut away if there is reason to believe that the excision will permit the construction of a more successful theory. We need not remain enslaved to concepts of classical theory, no matter how successful they are for macrophysics, if they are unobservable in the vastly different realm of microphysics.[23]

The Heisenberg principle shows us how to escape from the tight chains of the classical system of thought without danger of conflict with experience. If an aspect of classical theory is unobservable we are free to try to modify it or even to junk it altogether. Of course a declaration of freedom from unobservable classical concepts does not constitute a theory; it remains to be seen whether the new freedom conferred by the Heisenberg principle will help us to find a useful theory of microphysical processes.

The thoughts of the past several paragraphs have been qualitative. The Heisenberg principle contains, however, a quantitative aspect that has not yet been taken into account. The principle says much more than that "precise position and momentum cannot be observed simultaneously"; it tells us that the product of the uncertainties in position and momentum cannot be made smaller than $h \sim 6\cdot10^{-27}$ gcm^2/sec.[24] For electrons with energy

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[22] An extension of the example of the string figure provides a rough analogy. Obviously the figure cannot simultaneously have an exact "radius" and an exact "side length." If it has a radius then it does not have a side length. Suppose, however, that when a measurement of radius is made an interaction between the observing apparatus and the string occurs which forces the figure into a square. The side length is measurable on a circular figure even through prior to the measurement *the figure does not have the measured property*. In the course of measurement the figure gains a "side length" and loses a "radius."

[23] Nowadays we are accustomed to think so familiarly of atoms and electrons that we tend not to appreciate the enormous change in orders of magnitude that occur when we pass from the macrophysical to the microphysical realms. Normal macrophysical magnitudes are: length ~ 1 cm; mass ~ 1 g; time interval ~ 1 sec; charge ~ 1 microcoulomb. The factors which convert these macromagnitudes to proper microphysical magnitudes (for atoms) are 10^{-8} for length, 10^{-27} for mass, 10^{-16} for interval, 10^{-13} for charge! To get some impression of the difference a factor such as 10^{27} can make, note that 10^{27} meters is equal to 100 billion light years, a distance about ten times the size of the universe!

[24] This refers to *one* component of position and momentum. For each of the three orthogonal components the relation is the same. However, the x component of position and any *orthogonal* component of momentum are measurable simultaneously. Thus, if we locate an electron in a volume $\Delta V = \Delta x\cdot\Delta y\cdot\Delta z$, the uncertainties in the compo-

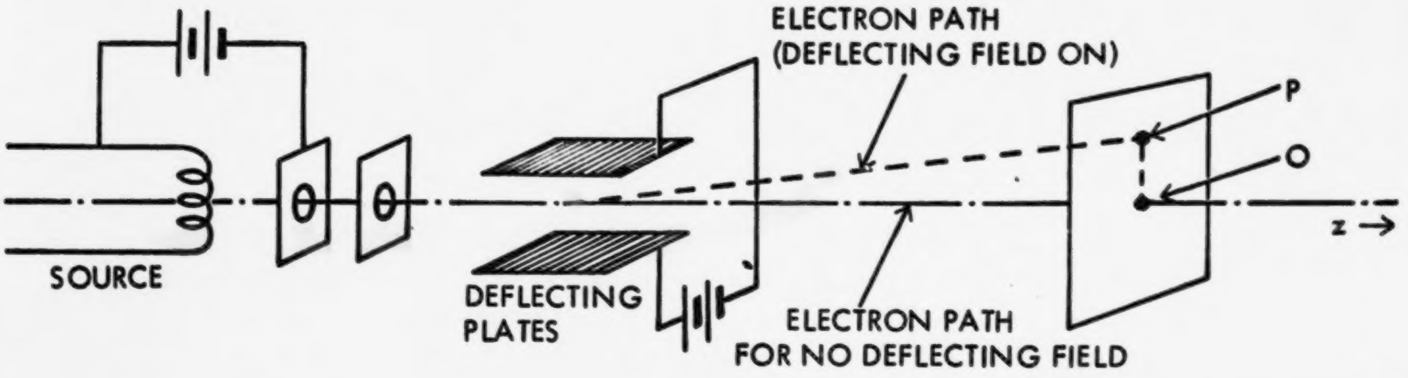

Fig. 6.4 Deflection of an electron beam by an electric field. In such experiments, classical concepts provide quite accurate predictions.

of the order of hundreds of electron volts (or smaller) and position uncertainties of the order of atomic dimensions, the restrictions imposed by the Heisenberg principle are very important; the classical notion of simultaneous position and momentum is unobservable even in the sense of a rough approximation. But consider an experiment on the deflection of electrons in an electric field by means of the apparatus shown schematically in Fig. 6.4. We may suppose the source and accelerating plates to be those used in the diffraction experiment (Fig. 4.1). A pair of plates between which an electric field may be generated is arranged so that when no field is present the electron beam passes between the plates and is detected at O. When the field is turned on the electrons arrive at P. The magnitude of the electric field and the geometry of the apparatus are chosen so that the deflection, OP, is several centimeters. In the discussion of the diffraction experiment it was shown that the momenta of the electrons in the beam are determined to within an uncertainty of about a half percent of the value of the momentum; the diffraction from 1 mm slits is negligible so that position in the x, y plane is determined to within a millimeter. For all the purposes of the observations to be made in the experiment the position and momentum are both quite well defined (i.e., percentage errors are small). Under these conditions the whole set of classical conceptions that are unobservable in the microscopic realm can be given fairly well-defined meanings. The path of the electron beam may be observed (naturally location measurements will be made with an accuracy $\Delta x \sim 1$ mm), the velocity and acceleration of the electrons may be measured with accuracy sufficient for our objectives, and Newtonian mechanics may be used to describe (approximately) the course of the electron motion.[25]

The foregoing complex of considerations may be brought together in a new interpretation of the Heisenberg principle. When the restrictions implied by $\Delta x \Delta p \sim h$ are effectively negligible (as in the example above) classical dynamical concepts have approximate significance; but in the realm of microphysics, i.e., where the restrictions of the uncertainty rela-

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nents of momentum Δp_x, Δp_y, Δp_z are such that $\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z \sim h^3$; i.e., $\Delta V(\Delta p_x \Delta p_y \Delta p_z) \sim h^3$.

The "phase space" (see Subsection 6.1) for an electron in three dimensions is a six-dimensional coordinate frame. In this "space" a point describes the three components of position and the three components of momentum of an electron. The Heisenberg principle states that location within a (six-dimensional) "volume" in phase space is unobservable if the "volume" is smaller than h^3.

[25] Many of the electron properties given in Section 3 are obtained by experiments in which classical conceptions are applicable.

tions are important - as they clearly are for electrons in atoms - classical modes of thought lose even approximate meaning. In particular, microphysical entities such as electrons do not have the property of exact simultaneous position and momentum. In the construction of a new theory of microphysical phenomena the Heisenberg principle may be used to free us from the constrictions of imagined but unobservable classical concepts. However the new theory must be capable of showing that classical concepts have an approximate validity wherever the restrictions imposed by $\Delta x \Delta p \sim h$ are unimportant. The role played by Planck's constant, h, in this interpretation is interesting. It provides a quantitative measure by means of which we judge whether a particular experiment or phenomenon may be interpreted classically (in some approximate sense), or whether we are free to try to develop a theoretical description of a nonclassical character.

7 DYNAMICAL PROPERTIES OF MICROSYSTEMS

> "Imagination is often
> at war with reason
> and with fact."
> —Benjamin Jowett

It is often difficult to make seemingly small changes in tightly organized structures. The alteration of a sentence in the middle of a paragraph may, infuriatingly, necessitate a complete reworking of several pages of material; similarly, changes in classical modes of thinking force modification of many interrelated classical concepts. The Heisenberg principle implies that microphysical phenomena are, in part at least, indeterminate (unpredictable in principle). The fact of indeterminacy undermines parts of the foundations on which classical physics is built and forces a re-examination of the classical concepts of "state" and "determinism."[26]

The considerations of Section 6 show that the Heisenberg principle may be interpreted in two markedly different ways: (1) the laws of nature make it impossible to measure some of the properties of microphysical entities; (2) microphysical entities do not have a number of the properties that observation together with classical assumptions imply. Measurement, in classical theory, simply discovers the properties that observed systems "objectively possess." Consequently, if an exact position measurement is always possible for an electron, an electron always has an exact location whether or not we know where it is; if, moreover, momentum is always measurable, then the electron always has both an exact position and an exact momentum. Subsection 6.3, however, examines the assumption that an electron never has an exact position and momentum simultaneously. This assumption implies that neither position nor momentum is always meaningful for an electron, even though either position or momentum is always measurable. Both the assumption of subsection 6.3 and its consequences are in conflict with traditional conceptions of the terms "measurement" and "property." It will be shown, however, that the classical interpretations of these words are based on metaphysical assumptions that are false in the realm of microphysics.

Naturally a break with customary usage leads to a host of questions. What is meant by "the electrons in this beam have a definite momentum," or, in more general terms, by "the system S has the property q'?" How are the properties of microphysical systems discovered? What, if anything, is meant by "a system has unobservable properties?" Before these questions can be considered, the presuppositions of the classical concepts of "property" and "measurement" must be exposed.

7.1 OBJECTIVE PROPERTIES.

Recall certain features of the classical use of the idea of a property. For example, as I write, I can't see Mars and, though I don't know where it is, I believe that Mars is "out there" somewhere. On what observational grounds is this belief justifiable? How can I know that Mars has a position if I don't know where to find it? Similar questions are often raised in beginning courses in philosophy. "How," the old chestnut goes, "do I know that the tree in my garden is still there when I'm not looking at it, or feeling its bark, or otherwise observing it? How do I know that the properties I see when I look at the tree are still 'out there' when I no longer give it attention? The very structure of these questions makes it impossible to establish by observation either of the statements: "all the properties of the tree remain unchanged even when I'm off on a trip," or "all properties of the tree disappear, or change in some unknown way when I'm not around to look at it." Despite my inability to establish the facts, however, I happily continue to believe that the tree placidly remains in my garden when I'm not observing it.[27] My attitude may be formulated more palatably, perhaps, as follows: I have a theory that is consistent with all my observations of the tree; it is the simplest consistent theory I know. For example: whether or not I look at the tree I know that on any sunny day I can find shade in its neighborhood; since the tree has some dead branches which may fall in a strong wind, I take care not to be in its neighborhood on windy days. In short, there is no aspect of my experience in conflict with the assumption that the tree is "out there" whether or not it is observed; i.e, the tree has what philosophers call an "objective existence." In essence the same remarks may be made in reference to our original question about the location of Mars. The assumption of the existence of "objective properties" may be justified, for macrophysical objects, in the same way that the objective existence of the tree is justified.

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[26] These concepts are analyzed in Section 8.

Our belief in the objective properties of stones, trees, dogs, etc., rests in part on the fact that they may be observed without altering their properties in the course of observation. Could I be so sure of the objective existence of my tree if after each momentary glance at it the tree sprouted a new branch, or withered, or changed its species? Probably not. Clearly our general belief in the objective properties of the entities of our experience is based on the assumption that these entities may be observed without modifying their properties.

But the arguments that justify the assumption of objectively existent properties are simply invalid in the realm of microphysics. An old joke says that no Irishman is so poor that there isn't another Irishman who can live well on his leavings. Similarly, classical physics assumes (not as a joke) that there is no physical system, however small, that there isn't another one with which the first may be observed without changing its properties. The preceding sections suggest that this assumption is false in the realm of microphysics. For example, an observation of position with an accuracy of 0.1Å on a free electron with a kinetic energy of 10 eV necessarily has cataclysmic consequences for the electron; the measurement may alter the energy by hundreds of electron volts. If the process of observing a tree produced changes such as a cyclone striking the tree might make, the processes of tree observation and accurate electron position measurement would have analogous features. A more subtle example of the modification of properties by the process of observation may be drawn from the diffraction experiments described in Section 4. Observation of whether each electron passes slit a or slit b (Fig. 4.7) cannot be effected without completely changing the two-slit pattern obtained when no observation of electron passage is made. Clearly, not all the measurable properties of microphysical entities can be observed "gently."

The argument based on the consistency of theory also fails. No adequate theory of the behavior of electrons is consistent with the assumption that the electron always has a location. In fact that assumption gets us into trouble by leading to the false expectation of a two-slit interference pattern consisting of a simple sum of two single-slit patterns. Since the assumption of an objective location property leads to difficulties and since, by the nature of the question, it cannot be known that an electron always has a meaningful location property, a continued insistence that the electron must always be at some well-defined place can only be interpreted as a form of masochism. Remember that with the unobservable creations of our imagination we are free to do whatever is useful or comforting.

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[27] In fact, unless one is a professional philosopher, any other belief carries with it the danger of getting oneself committed.

7.2 MEASUREMENT AND PROPERTY.

Consider a microphysical entity or system of some kind - an electron, a hydrogen atom, a methane molecule, etc. It is convenient to divide properties into two classes: (1) fixed or static properties, (2) dynamical properties. Fixed properties for electrons are electron mass and electron charge; a fixed property of the hydrogen atom is its composition - the system contains one electron and one proton. Once we know that we are measuring electrons, we know with certainty (because of the many measurements that have been made in the past) what the measurement of mass or of charge will yield. But, if all we know is that we are dealing with electrons, we cannot be certain of the result of a position measurement or a momentum measurement. Properties that may differ when measured on different systems of the same kind, or that may change in the course of time when measurements are made on a single system, are called dynamical properties. There are no problems of interest (for our present purposes) about the meaning of "fixed properties"; the following remarks relate to the class of dynamical properties. It is useful to take a wide view of what constitutes a property; in addition to such common quantitative properties as "the distance between electron and proton in a hydrogen atom is .5Å," "the energy of a free electron is 3.0 eV," more indefinite properties are sometimes useful: e.g., "the electron is to the right of a certain plane," "the x component of momentum is positive," "the distance between an electron and a proton is 1.0Å within an uncertainty of 0.5Å."

Because of the indeterminacies characteristic of microphysical phenomena it is helpful to take as the object of our consideration an ensemble of systems of the same kind (same set of "fixed" properties) rather than a single system. (Some of the reasons for the usefulness of ensembles have been noted in previous sections; others will appear as we go on [see section 7.5, Distributions]). Suppose now that we have an ensemble of systems of a certain kind; we indicate the kind of system contained in the ensemble by the symbol S. My object is to specify the meaning that is to be assigned to: "The systems, S, in the given ensemble, have the property q'."[28]

To give meaning to the quoted sentence there must be, first of all, a physical test for the property q' (of system S). The test will consist, in general, of some arrangement of apparatus, directions for arranging an interaction between the apparatus and the systems to be observed, and a specification of two classes of observable responses that result from the interaction of the apparatus and the observed system. The two response classes may be labeled "yes" and "no" (or "y" and "n"). We mean by a "test for property q'" (the test will be indicated by the symbol $\mathcal{T}(q')$) the apparatus, the method for arranging interaction and the specification of the "yes" and "no" responses. For example, the test for the property, "the x component of position of an electron is x' with uncertainty Δx," might consist of a microscope, a source of illumination, directions for location and orientation of the microscope and the sources, and the specifi-

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[28] In general discussions the symbols q', r', s', (or q'', q''', etc.) will be used to refer to dynamical properties of systems.

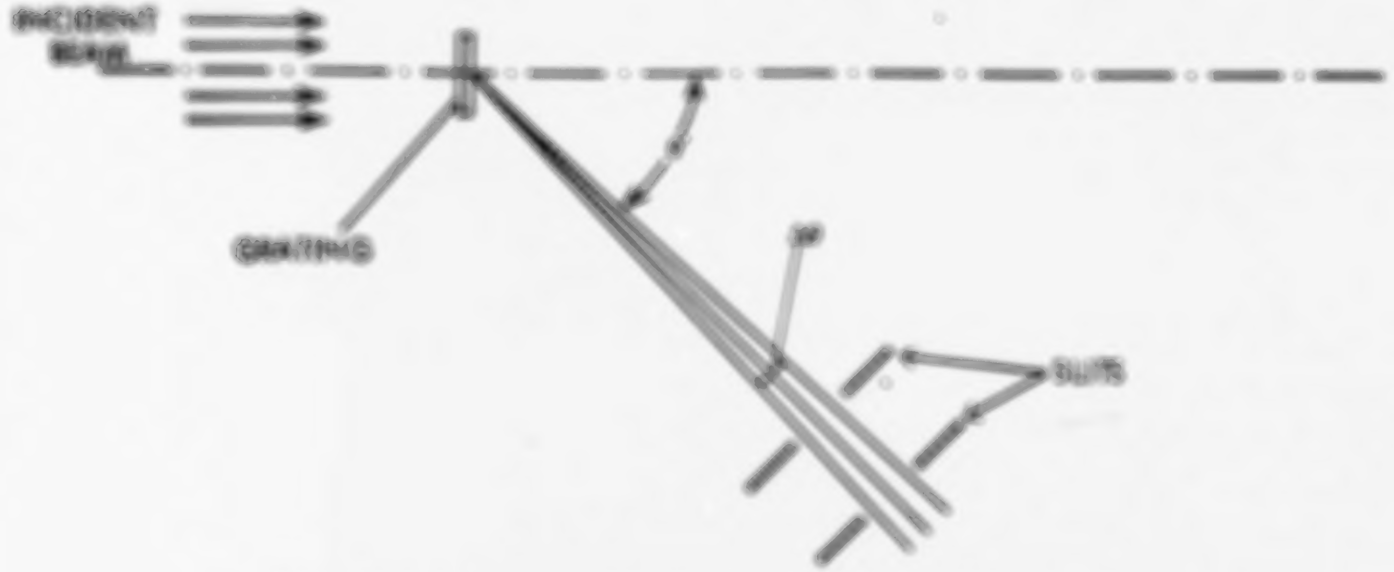

Fig. 7.1 A preparatory measurement for photons.

cation of the "yes" and "no" responses as the appearance of a scintillation within (yes) or outside (no) a specific area on a screen. The apparatus $T(q')$ forms a *part of the definition* of the property q'.

The system S in an ensemble will be said to *have* the *property* q' if, and only if, it is *certain* that the application of the test $T(q')$ to *every* S in the ensemble will result in a "yes" response. This definition is empty, however, unless it is possible to *produce* ensembles of S systems with property q', i.e., to prepare collections of S systems such that the interaction of each S in the collection with $T(q')$ results in a "yes" response. The process of producing an ensemble with the property q' will be called "preparation" or "preparatory measurement" of property q'. In general, a preparation also requires physical apparatus, methods for its disposition, and specification of classes of "yes" and "no" responses. For example, an ensemble of photons may be prepared with the property, "the energy of each photon is E eV within an uncertainty of 0.1 eV" by directing a beam of radiation at a grating (Fig. 7.1) and selecting, by means of a suitable arrangement of slits, only the photons that are deflected by a particular angle θ′ (within an uncertainty Δθ). Here the "yes" response is successful passage of the slits, the "no" response is failure to pass the slits.** An apparatus that prepares an ensemble with property q' will be designated by the symbol $\mathcal{P}(q')$. Both a preparation, $\mathcal{P}(q')$, and a test, $T(q')$, are essential to the determination of the *meaning* of the property q'. The property is undefined if no preparation, $\mathcal{P}(q')$, exists for a presumed test $T(q')$, or if no test exists for a presumed preparation, $\mathcal{P}(q')$. Note that a test process looks backward; there is no interest in the properties of the tested systems after the test is over. (The photoelectric effect, for example, could be used in a test for the energy of photons. In the test the photons are absorbed and no photon properties remain.) In the process of preparation, however, we look to the future; the property of the systems *after* preparation is of concern. For this reason the $\mathcal{P}(q')$ process is also called a "predictive measurement." If a system produces a "yes" response on

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**Quite frequently preparations in microphysics have characteristics similar to those of this example. An apparatus that prepares q' *eliminates* only systems that test negatively for q'; transmission is the "yes" response.

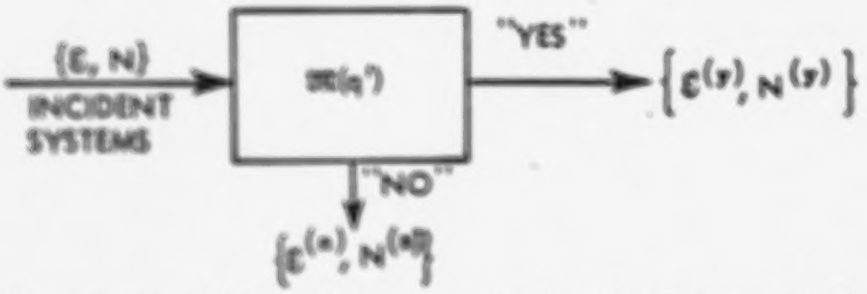
$\{\mathcal{E}, N\}$
INCIDENT SYSTEMS
$\mathcal{M}(q')$
"YES"
$\{\mathcal{E}^{(y)}, N^{(y)}\}$
"NO"
$\{\mathcal{E}^{(n)}, N^{(n)}\}$

Fig. 7.2 Diagrammatic representation of a measurement instrument $\mathcal{M}(q')$. The incident ensemble, $\mathcal{E}$, is separated into ensembles $\mathcal{E}^{(y)}$, $\mathcal{E}^{(n)}$ (N, $N^{(y)}$, $N^{(n)}$ are the numbers of systems in the ensembles $\mathcal{E}$, $\mathcal{E}^{(y)}$, $\mathcal{E}^{(n)}$); The $\mathcal{E}^{(y)}$ ensemble consists of systems that produced "y" responses in $\mathcal{M}(q')$. By definition the $\mathcal{E}^{(y)}$ ensemble has property q'. $\mathcal{M}(q')$ is also a test instrument for property q'; a system that produces a "y" response on interaction with $\mathcal{M}(q')$ is said to exhibit q' on test.

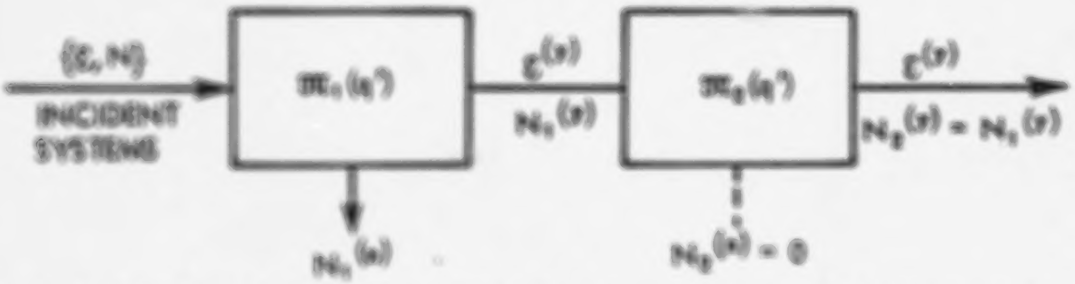

Fig. 7.3 Definition of property q' by means of $\mathcal{M}(q')$ instruments.

interaction with $\mathcal{T}(q')$ we shall say that $\mathcal{S}$ "exhibited property q' on test." Note that it is not necessarily implied that the system had property q' before the test was made. When a "yes" response occurs in a preparation $\mathcal{P}(q')$ the system is said to have been prepared with property q'.

Often the same instrument may serve both as a test and as a preparation of a property q'; an instrument of such versatility will be called a "measurement" and designated by the symbol $\mathcal{M}(q')$.* We shall assume that an $\mathcal{M}$ instrument exists for every definable system property. A useful diagrammatic representation of an $\mathcal{M}(q')$ instrument is shown in Fig. 7.2.

The consideration of the foregoing paragraphs may be summarized in the following definitions.

(a) "A system $\mathcal{S}$ may have a property q'" or "property q' is measurable on $\mathcal{S}$" means: There exists an instrument $\mathcal{M}(q')$ such that whenever $\mathcal{S}$ systems interact successively with two such instruments ($\mathcal{M}_1(q')$, $\mathcal{M}_2(q')$ - see Fig. 7.3) every $\mathcal{S}$ that produces response "yes" in $\mathcal{M}_1(q')$ also produces response "yes" in $\mathcal{M}_2(q')$; if N systems are incident on $\mathcal{M}_1(q')$ and $N_1^{(y)}$ systems produce response "yes," all these systems produce response "yes" on interaction with $\mathcal{M}_2(q')$.**

(b) "A system $\mathcal{S}$ with a given "history" has the property q'" means: In a large number of previous experiments it has been found that every system with the given "history" produces a "yes" response on interaction with a $\mathfrak{J}(q')$ (or $\mathfrak{M}(q')$) instrument; i.e., the specific "history" constitutes a *preparation* of property q′.

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*The definition of a property as formulated in preceding paragraphs in terms of $\mathcal{T}$ and $\mathcal{P}$ instruments is incomplete. The reader may enjoy discovering its ambiguities and supplying the details needed for its completion. The definitions in terms of $\mathcal{M}$ instruments, however, is unambiguous. Although $\mathcal{M}$ instruments are convenient for theoretical considerations, and are sometimes useful in the laboratory, most experimental investigations employ preparation ($\mathcal{P}$) and test ($\mathcal{T}$) instruments.

**Unless the property q' does not change with time, the interaction with $\mathcal{M}_2$ must follow immediately after the completion of the interaction with $\mathcal{M}_1$.

Several implications of these definitions are worth noting explicitly:

(1) The concept of an unobservable property has no meaning. A property is defined for a system only if instruments that prepare and test the property are given. Consequently the expression given to the Heisenberg principle in previous sections is no longer admissible. It is easily recast:

The Heisenberg Uncertainty Principle: For every real number x′ (or p_x') and every positive number Δx (or Δp_x) an electron may have the property, the x component of position (or momentum) is x′ (or p_x') within an uncertainty Δx (or Δp_x). There exists no preparation capable of producing an ensemble with both an x component of position (for any x′) with an uncertainty Δx and an x component of momentum (for any p_x') within an uncertainty Δp_x if the product $\Delta x \Delta p_x$ is less than a quantity of the order of Planck's constant, h.

(2) From the observation of a "yes" response in a test (or "measurement") of property q′, it does not follow that the system *had* the property q′ just prior to its interaction with the test apparatus;[32] the system may be said to have had property q′ before a test only if its history prior to the test constitutes a preparation of q′. Many errors of interpretation stem from a failure to distinguish between the meanings of "a system had property q′" and "a system exhibits property q′ on test."

(3) The meaning of sentences such as "the electron does not always have a location property" may be clarified. Suppose that experiments on many ensembles of electrons prepared with property q′ ("q′ ensembles") reveals that the electron position property, x′ (x component), is distributed over a range larger than D in every q′ ensemble.[33] In other words, in any q′ ensemble different values of x′ are found in tests on different electrons and the difference between the largest and smallest position values found is always greater than D. It follows from preceding definitions that in q′ ensembles, electrons do not have the property "location within a range," d < D"; q′ ensembles with the additional property, "location within range d," simply do not exist.[34]

The complex abstractions required for the general definition of "property" have surprisingly elementary exemplifications. Let's follow Newton in his discovery of the *property* of color. Newton was led, by considerations into which we need not delve (every student of physics must read his fascinating *Opticks*), to pass light from the sun through a prism and to observe its spectrum. No doubt large numbers of men had noticed the appearance of color when cut glass was illuminated. But Newton went a step further; he selected a small range of the colors produced by one prism (i.e., the radiation deflected by well-defined angle, θ′) by means of slits (Fig. 7.4) and passed this range of color through a second prism. In passing through the second prism the light was not spread out again as the original light from the sun had been spread into a spectrum by the first prism; instead the deflection in the second prism was again just θ′, the angle through which the selected radiation had been deflected originally. By

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[32] Past properties of systems are unobservable since the direction of time cannot be reversed. The variables employed in describing the unobservable past are unimportant except that they are often used as premises for improper predictions (see subsection 7.3.4).

[33] See Subsection 7.3 for a definition of the phrase, "distribution of position in an ensemble."

[34] E.g., in ensembles with momentum certainty Δp_x, electrons do not have a position property with uncertainty as small as [illegible]

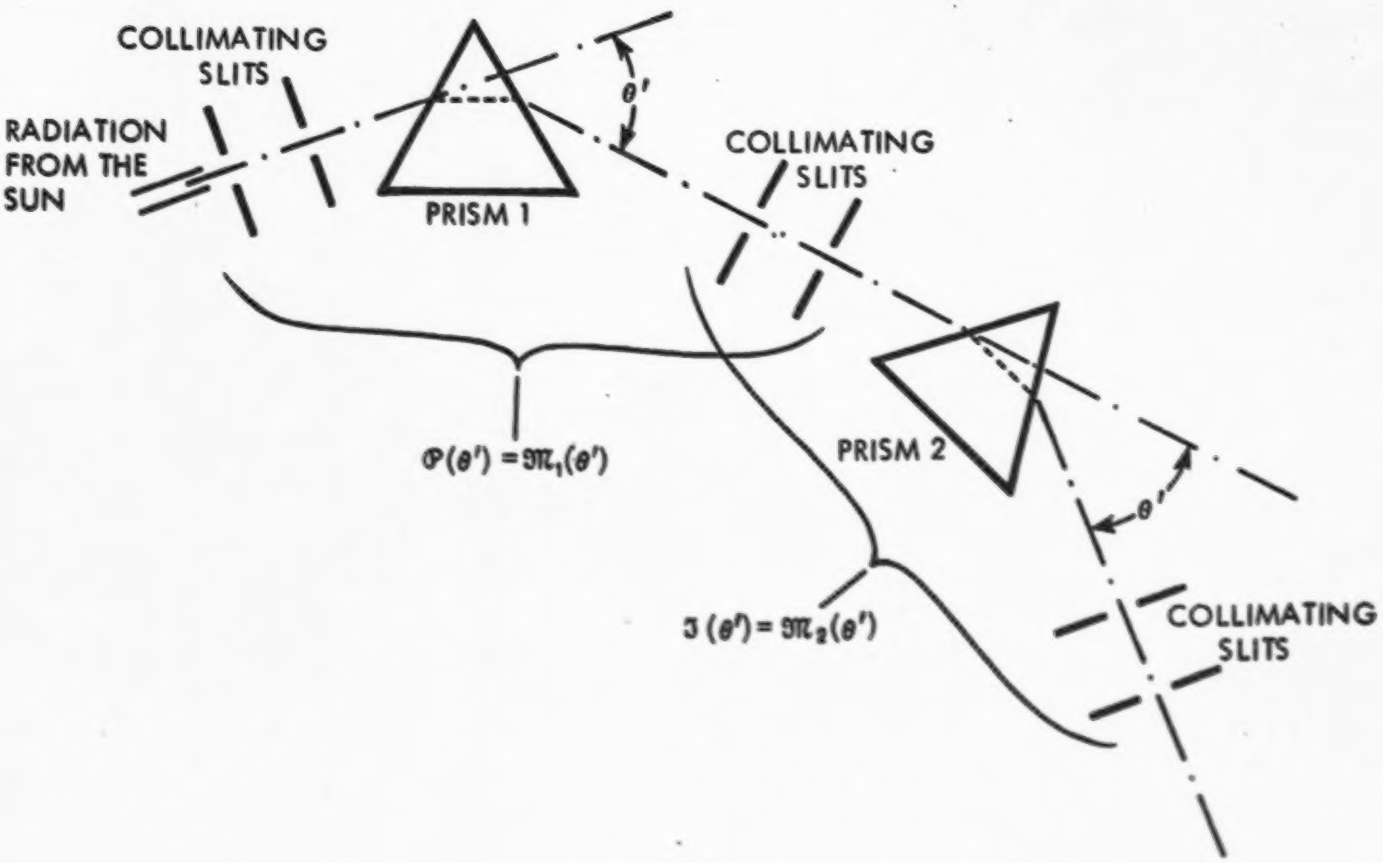

Fig. 7.4 Newton's measurement of color properties.

these observations Newton had discovered a property of light. The first prism (with the slit arrangement defining a deflection θ') constitutes the process of selection - the apparatus $\mathcal{P}(\theta')$; the second (with the same slit arrangement) is the test, $\mathcal{T}(\theta')$. Both $\mathcal{P}(\theta')$ and $\mathcal{T}(\theta')$ consist of the same equipment, i.e., the arrangement is also an $\mathcal{M}(\theta')$. The "yes" response in both is passage through the slits defining deflection θ'. Newton discovered that any radiation with a "yes" response in $\mathcal{M}_1(\theta')$ leads to a "yes" response in $\mathcal{M}_2(\theta')$ (see Fig. 7.4). It is the whole set of these observations which permits us to speak of a property of radiation; the property in question may be formulated as "deflection by θ'," or by the associated "color" as judged by eye, since the deflection angle and color are correlated.

The complexity of the definition of a dynamical property indicates that the discovery of properties is no simple matter. As an example of the difficulty, let's try to find a dynamical property of electrons by imitating Newton. Suppose a beam of electrons is produced by (say) the procedures used in diffraction experiments and directed on a film of material which transmits and scatters the incident electrons. From the scattered electrons those deflected by an angle θ' are selected for consideration. Do the electrons so prepared have a "property" distinguished by their deflection through θ'? The answer is yes if an $\mathcal{M}$ apparatus exists such that all electrons scattered through θ' produce "yes" responses in $\mathcal{M}$. Since a measurement, $\mathcal{M}$, with the desired property is not known, the preparation does not determine a property of electrons.[**]

It is the business of experiment to search out all observable properties of a system and to discover all the correlations among them. Thus, for example, it is found that photons with the property, "deflection, θ', by a particular prism," also have definite energy; therefore, the property, "deflection by θ' in the prism," implies the property "a definite energy E'," or "measurement of θ' by the prism" is equivalent to a "measurement of the energy E'." A very different type of correlation is frequently met. In the diffraction experiment we first produce an ensemble of electrons with a well-defined momentum property (Fig. 4.1). The fluorescent screen (or photographic plate) is a device which tests for a range of position properties on those electrons of the ensemble that get through the slits placed between the source and the screen. Electrons (prepared with the same momentum) are found at different locations on the screen. We do not, therefore, have a unique correlation between the initial momentum property and a tested position property. However there is a well-defined correlation between the initial momentum and the distribution in the tested positions, i.e., between the initial momentum and the diffraction pattern. Indeed, most of the correlations that are discovered in the realm of microphysics are of this general character. It is important to recognize that the position distribution does not have meaning for a single electron with definite momentum but is a characteristic of an ensemble of electrons prepared initially with definite momentum.

[**] Other examples of the experimental definition of microphysical properties will be examined in Chapter 8.

Theory must provide a set of general rules by means of which the observable correlations among properties may be deduced. Thus, theory should be able to predict the correlation between deflection, θ', (in a prism) and photon energy, E', or to predict the correlation between incident momentum and the distribution in position that appears on a photographic plate in a diffraction experiment (given the geometry of the apparatus). However, theory need not describe imagined but unobservable features of any process; theory need not tell us, for example, how the ensemble of electrons in the incident beam gets through the slits in the process of forming a diffraction pattern.

7.3 INCOMPATIBILITY

The Heisenberg principle states that it is impossible to prepare an ensemble with position and momentum uncertainties (Δx, Δp_x) such that $\Delta x \Delta p_x$ is less than h (approximately). This principle suggests (Section 5) that many other pairs of properties - e.g., location of the electron in a hydrogen atom, within $\Delta x \sim .1$Å and an exact energy property of the atom - are not simultaneously measurable. We have not yet had occasion to mention microphysical properties that can be measured simultaneously. Examples are easily discovered. An electron may be located in space and this location may be converted into components along three chosen axes; thus the components x', y', z' of the position of an electron are simultaneously measurable. Similarly, the direction and magnitude of the momentum of an electron may be measured (e.g., by the arrangement employed to produce the incident beam (Fig. 4.1) in the diffraction experiments); from the direction and magnitude, the three perpendicular components of momentum along the axes of a chosen coordinate frame are easily obtained. Less trivially, the x component of position and y component of momentum may be measured together. [Problem.] A somewhat different example - not quite as trivial as it may seem on first sight - is afforded by the pair of properties, "position located in a segment Δx" and "position located in a segment δx", where δx is inside the Δx segment (Fig. 7.3). A measurement of the property "location in δx" is simultaneously a measure of "location within Δx." For any electron with the "δx" property I am certain

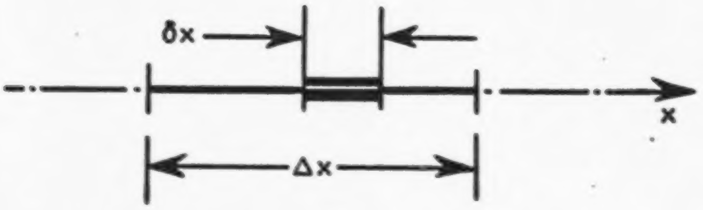

Fig. 7.5 An example of simultaneously measureable properties.

to get a "yes" response if I test for the "Δx" property[36] (i.e., if I am certain of finding, on test, a location within δx, it is certain that I shall find, on test, location within Δx). Properties which can be measured simultaneously are said to be compatible. The properties q′, r′, of system $\mathcal{S}$, are compatible if it is possible to prepare an ensemble of $\mathcal{S}$ systems that have simultaneously the properties q′, r′; this means that it is certain that if a test for q′ is made by $\mathcal{T}(q')$ a "yes" response is obtained and a similar statement is true of tests for r′.

The existence of compatible properties is hardly surprising; classical ideas lead us to expect compatibility. For a Newtonian particle, exact position and exact momentum are compatible; in classical theory, electric and magnetic fields are compatible. If (using classical assumptions) a system, $\mathcal{S}$, has a property, q′, and on observation a property, r′, is observed, then properties q′ and r′ are compatible; for classical theory tacitly assumes that the measurement of r′ on a system with property q′ can be made with such care as not to change q′; after the r′ measurement (preparation), the system has properties q′ and r′. (We assume that the r′ measurement is made immediately following the measurement of r′.)

When an ensemble of systems prepared in some manner is such that on test for q′ no system produces a "yes" response it is said that the ensemble does not have the property q′. A property q″ is said to be exclusive to (or to exclude) a second property q′ if all ensembles with property q″ do not have the property q′, i.e., whenever an ensemble has property q″ it is certain that q′ will not be exhibited on test. (Naturally q′ is assumed to be a measurable property of the systems in the ensemble.) It appears to be a physical fact that whenever a property q″ is exclusive to q′ then q′ is exclusive to q″ - the relation of exclusiveness is symmetric. Thus we may speak of q′ and q″ as an exclusive pair of properties. The relation of exclusiveness is well known both in classical physics and in microphysics. Two different electron position properties are exclusive; if an electron is at a position x′ it is certain that it is not at x″ if x″ ≠ x′. Similarly, two different energy values (of a hydrogen atom, say) are exclusive. There are less obvious examples: the "total energy E′" and "kinetic (or potential) energy greater than E′" are exclusive in classica theory; for systems consisting of radiation samples, any two polarizatio properties are exclusive in classica theory (in microphysics these proper ies are not exclusive in general - se below).

In classical theory, measurement without disturbance of th properties of the measured system is assumed to be possible. On systems wth property r′, any second property s is either never found on measuremen or sometimes found. If "never" ten r′, s′ are exclusive. If "sometimes," then there exist systems with oth r′ and s′ properties, i.e., the ′, s′ properties are compatible. Clarly, then, any two properties (in classical theory) are either compatibl or exclusive.

For microphysical sytems, however, there are pairs of roperties that are neither compatibe nor exclusive. Certainly a position x′ of an electron and a momentum p are not compatible; by the Heisenberg principle it is impossible to prepare an electron with both exact position and exact momentum properties. But neither are these properties exclusive. On an ensemble of electrons at a definite position some electrons with a definite momentum p' may be found on test. Pairs of microphysical properties that are neither compatible nor exclusive will be called incompatible. Any exact position property is incompatible with any exact momentum property. It is easily seen from the symmetry of the relation of exclusion that incompatibility must also be a symmetric relation; if q' is incompatible with r', then r' is incompatible with q' and we may speak of a pair q', r' of incompatible properties. [Problem]

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[36] This example indicates the nature of the ambiguity referred to in footnote 34. On an ensemble with property, "location in δx," tests for the property, "location within Δx," are certain to produce "yes" responses although this is not a test for the property possessed by the ensemble.

The fact that pairs of microphysical properties may be incompatible - neither simultaneously measurable nor exclusive - constitutes the central difference between classical expectation and microphysical experience. From the chapters to follow it will be seen that incompatibility is a far more frequently met relation between properties than is compatibility. Some examples of incompatible properties were indicated in Section 6. No exact electron position property is compatible with an exact energy property E' for the hydrogen atom (classically

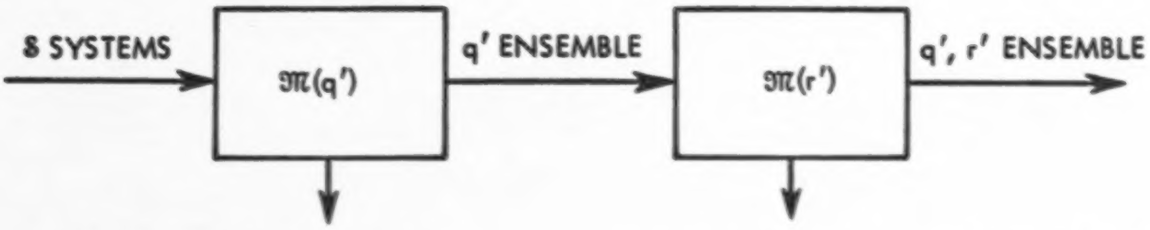

Fig. 7.6a Compatible Properties. The $\mathfrak{M}(q')$ and $\mathfrak{M}(r')$ instruments may be interchanged. The ensemble transmitted by the second instrument--systems with "yes" responses in both instruments--has both q' and r' properties.

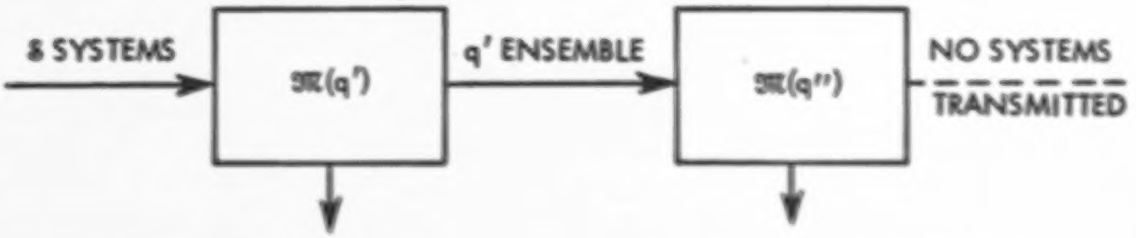

Fig. 7.6b Exclusive Properties. None of the systems in the q' ensemble produces a "yes" response in $\mathfrak{M}(q'')$.

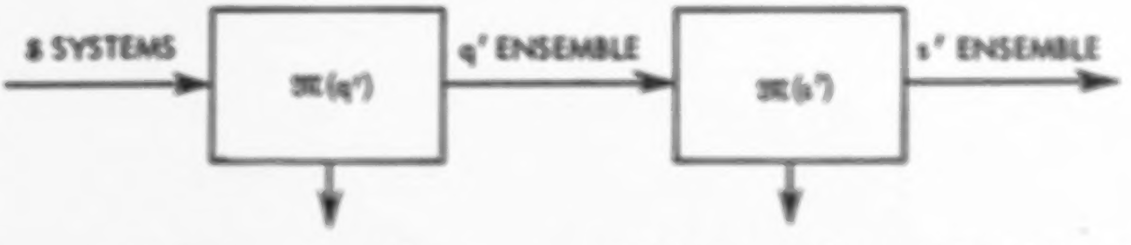

Fig. 7.6c Incompatible Properties. Some, but not all, of the systems in the q' ensemble produce "yes" responses in $\mathfrak{M}(s')$ and form an s' ensemble. However, this ensemble does not have the property q'.

Fig. 7.6 Diagrammatic representations of the meanings of compatible, exclusive and incompatible properties.

there is a set of position properties compatible with E′ and a second set exclusive to E′). Any two components (along distinct axes) of the angular momentum of a system are incompatible (classically they are compatible). The properties "total energy E′" and potential energy greater than E′" are incompatible (exclusive classically) in microphysical systems. The pairs x′, p′ (position, momentum properties) are compatible for any value of x′, p′ in classical physics, and always incompatible in microphysics. In Chapter 2 it will be shown that different polarization properties (of radiation), which are treated in classical physics as exclusive, are generally incompatible; special pairs of polarization properties, however, are exclusive (two linear polarizations polarized in orthogonal planes are exclusive, as are also left and right circular polarizations.

The three types of relations - compatible, exclusive, incompatible - that have been discussed above are represented diagrammatically in relation to measurement instruments in Fig. 7.6.

7.4 OBSERVABLES.

In the laboratory, tests for specific properties, such as energy E′ or position in Δx about x′, etc., are less common, perhaps, than experiments designed to "measure" the "energy," or "position," etc., of an entity. It is necessary to understand what is meant by "energy," "position," in such expressions and to know what is implied by their "measurement."

Usage is not firmly fixed, but, if no qualifications are specified, "an energy (or position, etc.) measurement on a system" generally means a simultaneous test for all the exact energy (positions, etc.,) properties the system may have. The "energy" of electrons in a beam might be "measured," for example, by the apparatus illustrated in Fig. 7.7. An electron incident on the photomultiplier produces a pulse that is displayed on the cathode ray tube of the oscilloscope; from the "height" h′ of the observed pulse a unique energy value E′ is determined. It is useful to consider this instrument as a test apparatus for any of the energy properties that it "measures." The instrument has the following characteristics: if an ensemble prepared with energy E′,is incident upon the apparatus, each electron produces a pulse of height h′; electrons in any ensemble with a property exclusive to E′ never produce a pulse of height h′. In the language of subsection 7.2, height h′ is the "yes" response, and the set of heights other than h′ is the class of "no" responses for the property E′. Each of the responses of the instrument - each height - is a "yes" response for one of a set of exclusive properties, the different energy properties of electrons.

The foregoing example illustrates the general characteristics of an instrument that can test simultaneously for a set of exclusive properties,- say the set of n properties, $\{q^{(i)}$,

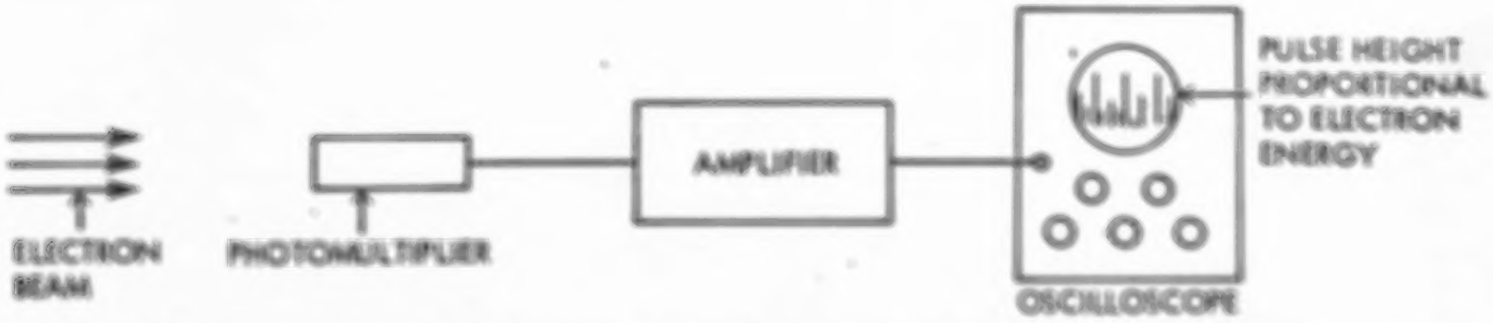

Fig. 7.7 An instrument for testing a range of the energy properties for electrons.

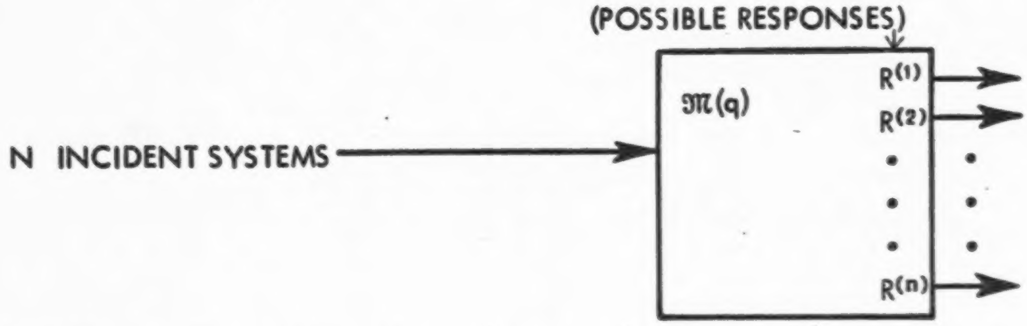

Fig. 7.8 Schematic representation of a measurement instrument for an observable q consisting of a complete set of exclusive properties. $q^{(1)}$, $q^{(2)}$......$q^{(n)}$.

$q^{(2)}, \cdots q^{(n)}\}$. The instrument must be capable of n + 1 responses (or response classes) $R^{(1)}, R^{(2)}, \cdots R^{(n)}, R'$, where $R^{(k)}$ is the "yes" response for the property $q^{(k)}$ and all the other responses comprise the "no" class for this property (k may take any value from 1 to n). The response R′ to a system means that none of the properties of the set $\{q^{(1)}, q^{(2)} \cdots q^{(n)}\}$ is exhibited.[37] As with the tests described in subsection 7.2, the observation of a response $R^{(i)}$ does not in itself imply that the system tested had (before the test) the property $q^{(i)}$.

It has been noted that the words "energy," "position," etc. (in "energy measurement," "position measurement," etc.) refer to sets of exclusive properties - all exact energy properties, all exact position properties, etc. These sets of properties are called "observables"; the set of all energy properties E′, E″, ... of a system is the "energy observable," E, for the system; similarly, the set of all position properties (of an electron, say), x′, x″, ... is the position observable, X, of an electron.[38] The property sets belonging to observables have two characteristic features: (1) the different properties in the set are mutually exclusive; (2) if all the properties in the set constituting an observable are tested simultaneously on a system some one property in the set is certain to be exhibited (e.g., if tests for all possible positions are made simultaneously on an electron, the electron is certain to be "found" somewhere). Any set of properties that satisfies condition (2) is said to be "complete."

The foregoing considerations suggest a general definition: any set of properties - say $q^{(1)}, q^{(2)}, \ldots$ - of a system that are (1) mutually exclusive and (2) complete, is called an observable, q, of the system. An instrument that can test simultaneously for all the properties in observable q will be called a "test of observable q" and denoted by $\mathfrak{J}(q)$.[39]

If a test instrument $\mathfrak{J}(q)$ for an observable, q, is such that after a test exhibiting any one of its properties - say $q^{(i)}$ - the tested system has the property $q^{(i)}$, the instrument will be said to be a measurement of q and will be denoted by $\mathfrak{M}(q)$. Such an instrument not only can test for q, but can be used to prepare systems with any of the properties in the observable. Figure 7.8 provides a diagrammatic representation of an $\mathcal{M}(q)$ instrument.[40]

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[37] A photographic plate, for example, tests simultaneously for the set of positions at which the active grains of the plate are located. (What is R′?)

[38] In classical physics it is customary to call such sets of properties "variables" (e.g., "energy variable," "position variable").

[39] Figure 7.7 illustrates a $\mathfrak{J}(E)$ instrument for the observable E (energy of electron). Examples of observables other than those already mentioned are easily invented; the pair of properties "position to the left of a given plane" and position to the right of, or on, the plane constitutes an observable. In Chapter [illegible] polarization observables (for photons) will be examined. Any two exclusive polarization properties, e.g., right and left circular polarizations, form an observable. [Problems]

Two observables q, r, are called "*compatible observables*" if the pairs of properties $q^{(i)}$, $r^{(j)}$ ($q^{(i)}$ any property of q, $r^{(j)}$ any property of r) are *either compatible or exclusive*.[41] (For *any* two observables, r, s, not all the pairs, $r^{(i)}$, $s^{(j)}$, can be exclusive. Why not?) Two compatible observables, q, r, uniquely determine a third observable (denoted by "q & r") consisting of the set of all the *compatible* pairs $q^{(i)}$, $r^{(j)}$, that may be formed from q and r. Two observables, say q, s, are said to be "*incompatible observables*" if among the pairs $q^{(i)}$, $s^{(j)}$, at least one is an imcompatible pair. Obviously the position and momentum observables are incompatible; in this case *no two* position and momentum properties are compatible. Momentum and any polarization observable (for photons) are compatible; indeed, every momentum property is compatible with every polarization property, i.e., photons may be prepared with *any* definite momentum *and any* polarization property. Two polarization observables, however, are incompatible; this fact follows from the observation that it is impossible to prepare photons which have two polarization properties simultaneously (see Chapter 2).

In *classical* physics the "dependence of an observable on time" is frequently used to describe the "motion" of a system; e.g., a motion of a free particle in one dimension is described by a function, x(t), which specifies the "position" of the particle at each instant, t. Although the meaning of x(t) is clear, the phrase, "dependence of an observable on time," is ambiguous. An observable is a set of properties (complete and exclusive) for systems of a definite kind, and the definition of an observable makes no reference to time. Therefore, the phrase quoted above is nonsense if interpreted literally. An observable has meaning for all systems of the same kind while functions like x(t) refer to *a* particular system in *a* particular motion. The function x(t) tells us what property, in the set constituting observable x, the particular system under consideration has at any definite time; at t′ the system has property x′, at t″, the property x″, etc., where, at the different times, we are concerned with the properties in the observable x. We shall call a function, $q_{\mathcal{S}}(t)$, which tells us the property in q that the system $\mathcal{S}$ has at time t, for a range of different times, the "motion of q" in $\mathcal{S}$ (often we shall use q(t) instead of $q_{\mathcal{S}}(t)$, but the fact that the function refers to a particular system, rather than systems of a certain kind, should be kept in mind).

The system properties that have been considered so far are properties at some time instant. It is possible *in classical physics* to define properties of a more complex character. Consider, for example, (for classical "particles") the property, "position x′ *and* position x″ t seconds later." A whole motion, as described by x(t), is also a system property. Such properties may be tested and prepared. The test for a motion x(t) consists of a succession of tests of the position observable at a large number of closely spaced intervals. Also, as is known from classical mechanics, if a motion, x(t), is observed on a particle, other particles of the same kind can be prepared - by selecting the position x(0) and the velocity v(0) that the motion x(t) has at t = 0 - so that all the suitably prepared particles have the *same* motion.

In microphysics, however, *motions of observables are not*, in general, *system properties*. Consider an electron. Suppose that the position observable is measured at some instant, $t = t^0$, and again at t′. The position properties exhibited, say, are found to be x^0, x'. By a large extension of the number of measurements over a succession of instants, the function $x(t)$ can, in principle, be tested, but let us restrict ourselves for the moment to the consideration of the pair of properties: x^0 at t^0, and x' at t'. Is this "composite property" a system property? The test of the putative property is clear; it consists of $\mathfrak{J}(q^0)$ at t^0 and $\mathfrak{J}(q')$ at t'. (Only the interval, $t' - t^0$ - not the specific times, t^0, t' - is of significance.) Beside the test, however, a preparation is required. But electrons cannot in general be prepared so that a test $\mathfrak{J}(q^0)$ at t^0 and $\mathfrak{J}(q')$ at t' are both certain to succeed ("yes" responses in both tests). We can prepare property q^0 at t^0. To arrange that q' at t' is also certain the electrons with property q^0 must be selected for some additional property that is compatible with x^0. In classical physics the additional property is the velocity, or momentum. An additional property for electrons that is compatible with x^0, such that when x^0 and the additional property are prepared, the electron is certain to test for x' at t', does not exist. If, now, we consider a motion - a large set of x properties at a large number of instants - the problem is greatly magnified. In general, microphysical systems cannot be prepared so that it is certain that they will execute specific motions. This simply reflects the observation that the behavior of microphysical systems is indeterminate.

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[40] A spectroscope is an example of a measurement instrument for the observable, "energy of photons."

[41] Our definitions imply that all classical observables (variables) are compatible.

Microphysical and classical theories, clearly, must have very significant differences. In classical physics, theory attempts to derive the motions of observables, i.e., functions such as $x(t)$, from simpler system properties (the position and velocity at $t = 0$, say) and general laws. Microphysical theory has no interest in "motions of observables" since such motions are not system properties; functions, such as $x(t)$, which play so large a role in classical physics, are without precise meaning in microphysics.

7.5 DISTRIBUTIONS

Because of the indeterminate characteristics of microsystems the objects of most interest in microphysics are not single systems but ensembles of systems. In effect "indeterminacy" means that regular, repeatable behavior in single systems is not found. Ensembles are used to investigate relations among the statistical properties of systems. Examples of interesting statistical regularities have been noted: in diffraction experiments the "distribution in position" of the detected electrons forms a definite pattern; in radioactive decay of neutrons the fraction of the number of neutrons remaining after time t is a definite exponential function of t. Clearly, ensembles and their statistical properties are of importance for microphysical investigations.

The central concept in statistics is that of the distribution of an observable. For the study of microphysics it is necessary to understand not only the meanings of microphysical "property" and "observable" but also the general characteristics of distributions in microphysical ensembles. Because of the differences between classical and microphysical concepts of property, the meaning of a "distribution" for microphysical ensembles is not quite the same as it is for classical ensembles.

7.5.1 Distributions in Classical Ensembles.

A distribution is defined relative to a particular ensemble and a particular observable - say $\mathcal{E}$ and q. Examples are: (1) the distribution in "height" (observable) in the "set of all twenty-year old males in New York State" (ensemble); (2) the distribution in "heads, tails" (this pair of properties forms the observable) in the "set of a thousand tosses of a penny" (ensemble); (3) the distribution in position (observable) of the "molecules in a container at time t′" (ensemble).[42] To avoid needless complication it is useful to suppose that the observable, q, with which we are concerned is "discrete," i.e., that the observable consists of an enumerable set of properties - say $q^{(1)}$, $q^{(2)}$, ..., $q^{(n)}$, ...[43] (The set may be finite or infinite.) Let N be the number of systems in $\mathcal{E}$ and $N^{(k)}$ the number of systems in $\mathcal{E}$ with property $q^{(k)}$. The number designated by the symbol $\{q^{(k)}|\mathcal{E}\}$ is defined by:

$$\{q^{(k)}|\mathcal{E}\} \equiv N^{(k)}|N = \text{fraction of the systems with property } q^{(k)} \text{ in } \mathcal{E}. \tag{7.1}$$

Note that

$$\sum_k \{q^{(k)}|\mathcal{E}\} = \sum_k N^{(k)}/N = \frac{1}{N}\sum_k N^{(k)} = \frac{N}{N} = 1, \tag{7.2}$$

where the sum over k is extended over all the properties in q. The set of fractions $\{q^{(k)}|\mathcal{E}\}$ for k = 1, 2, etc., specifies the "distribution of q in $\mathcal{E}$."

The set of numbers $\{q^{(k)}|\mathcal{E}\}$ for k = 1, 2, ... n ... may be considered as a function defined on the domain consisting of the properties in the observable q; the ensemble determines, for each $q^{(k)}$, a unique number, $\{q^k|\mathcal{E}\}$. The function so defined is the distribution function for q in $\mathcal{E}$; this function will be denoted by $\{q|\mathcal{E}\}$; the value of this function at the property $q^{(k)}$ is $\{q^{(k)}|\mathcal{E}\}$. Thus the "distribution of q in $\mathcal{E}$" and the "function $\{q|\mathcal{E}\}$" have the same meaning; both specify the association of the number $\{q^{(k)}|\mathcal{E}\}$ to the property $q^{(k)}$ (k = 1, 2, etc.).

The number $\{q^{(k)}|\mathcal{E}\}$, the fraction of the systems in E with property $q^{(k)}$, is also the probability that a system chosen at random from $\mathcal{E}$ has property $q^{(k)}$. (Clearly the probability of choosing any particular system from the ensemble is 1/N; since $N^{(k)}$ systems in $\mathcal{E}$ have property $q^{(k)}$, the probability that the system has $q^{(k)}$ is $N^{(k)}/N = \{q^{(k)}|\mathcal{E}\}$.) Therefore, the function, $\{q|\mathcal{E}\}$, is also called the probability distribution of q in $\mathcal{E}$. (Examples of several $\{q|\mathcal{E}\}$ functions are displayed, in graphical form, in Fig. 7.9.)

Fortunately it is not necessary to measure the observable q on all the entities (systems, objects) in a large ensemble, $\mathcal{E}$, to obtain the distribution $\{q|\mathcal{E}\}$. No one would measure, for example, the heights of all twenty-year-old males in the United States in order to obtain a distribution function. It is sufficient to make measurements on a relatively small, but randomly selected, "sample" of systems from $\mathcal{E}$ (the sample is a "subensemble" of $\mathcal{E}$; we denote it by $\mathcal{E}_s$). The distribution function, $\{q|\mathcal{E}_s\}$, obtained by measurement on the sample, will not, in general, be equal to the "true" distribution $\{q|\mathcal{E}\}$. By taking a sufficiently large sample, however, the difference between the true and sample distributions can be made as small as we like.* Even when the sample is "suf-

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[42] Note that an ensemble may change in time as in example 3; time has no significance, however, in relation to example 2.

[43] "Continuous" observables such as "position," "momentum," "height" (in example 1) will be considered in Chapter 3. The restriction to discrete observables is not a very significant one. A continuous observable containing a continuous range of properties is always an idealization and cannot be measured with complete precision. If the x component of position, for example, is measured with an accuracy of about Δ, the observable that is measured is composed (in a rough sense) of a set of nonoverlapping intervals of size Δ covering the whole x axis. This observable is discrete.

*Naturally the distribution function obtained by measurement on a "small" sample may be quite different from the distribution function of the given ensemble. The errors are small only in a statistical sense; the probability that a distribution measured on a sample differs from the true distribution by more than some specified error can be made as small as we please by making the sample ensemble sufficiently large.

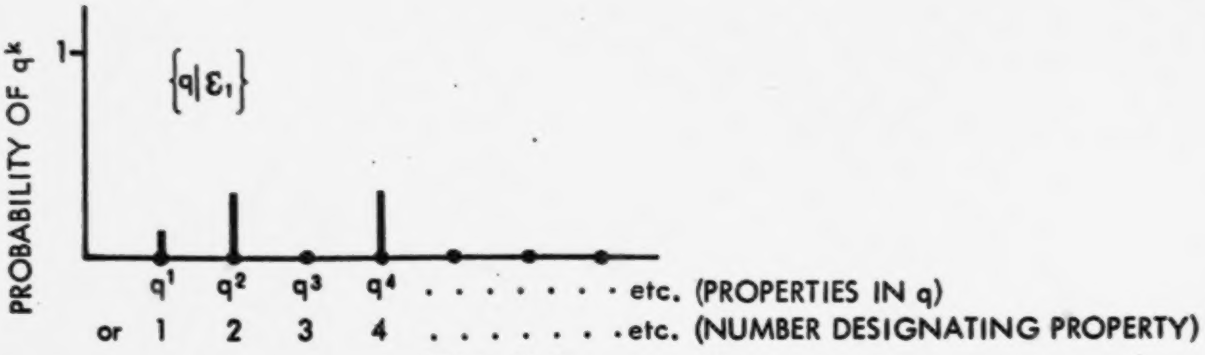

Fig. 7.9a

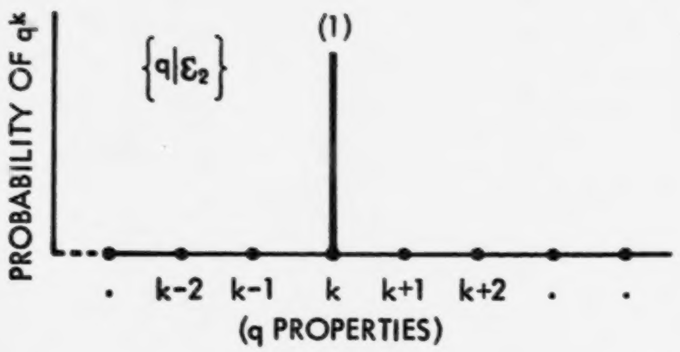

Fig. 7.9b

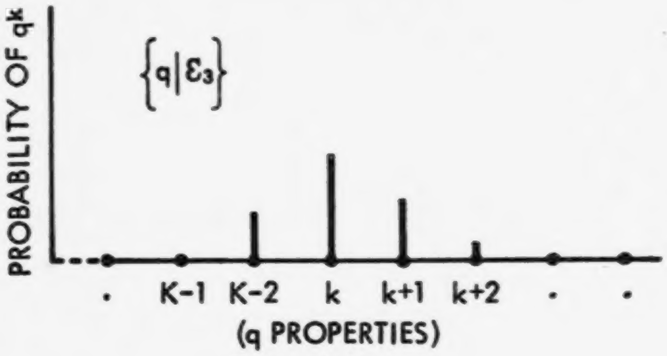

Fig. 7.9c

Figs. 7.9 a,b,c. Plots of distributions of q in various ensembles, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$. The distribution of q in $\mathcal{E}_2$ is said to be sharp.

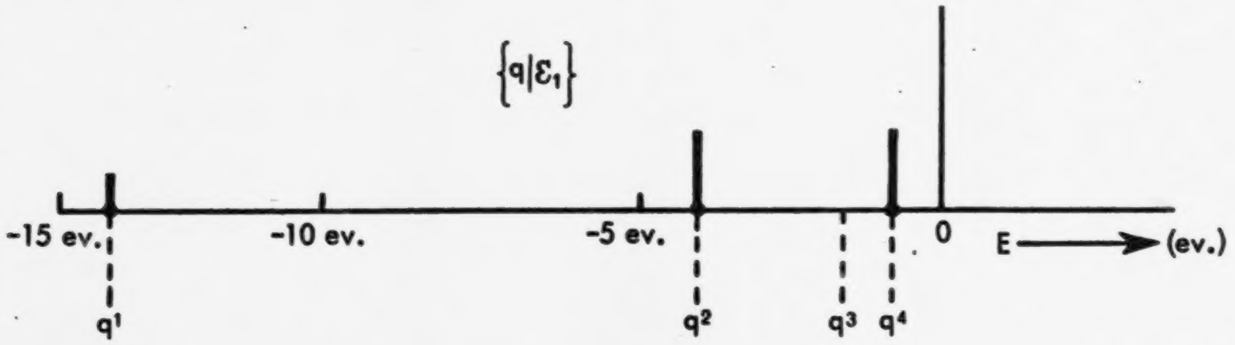

Fig. 7.9d For quantitative properties the distribution is generally given as a function of the numerical measure of the property. The $\{q|\mathcal{E}_1\}$ of Fig. 9a is plotted on the assumption that $q^{(k)}$ is the property "energy = -13.6 ev/k^2.

ficiently large," it may still be only a small part of the original ensemble. (We shall not be concerned here with the question of how large a sample must be to achieve a particular level of accuracy in the distribution.)

7.5.2 Distributions in Microphysical Ensembles

The development in the preceding paragraphs cannot be directly applied to microphysical ensembles. In general, and particularly if nothing is known about the preparation of the ensemble, no clear meaning can be given to the idea that a system in the ensemble has one or another of the set of properties in a microphysical observable, q (see subsections 7.1, 7.2). To be sure, a test for q on a system in $\mathcal{E}$ is certain to "exhibit" one or another of the properties in q, but this does not imply that the system had the property it exhibits (on test) before the test was made. If all the systems in $\mathcal{E}$ are tested, the number $N^{(k)}$ of the systems that exhibit $q^{(k)}$ may be counted for each of the properties in q, and these numbers may be used to define a distribution function as in the classical case. But what use can be made of the function so obtained? Just after the test we are left with an ensemble quite different, in general, from the one just before the test. Though the number $\{q^k|\mathcal{E}\}$ is obviously the probability of observing in $\mathcal{E}$ the property $q^{(k)}$, on test, this information is of no use since the original ensemble ($\mathcal{E}$) *no longer exists*. (Of course the systems remain; $\mathcal{E}$ has been changed into an ensemble with different properties.)

However, these difficulties are avoided easily. A *sample*, $\mathcal{E}_s$, consisting of a small fraction of the ensemble $\mathcal{E}$, is selected at random; even though $\mathcal{E}_s$ is only a small part of $\mathcal{E}$ we assume that the number of systems, N_s, in $\mathcal{E}_s$ is "large" (the meaning of "large" will be explained a little later). By test for q on $\mathcal{E}_s$, the number, $N_s^{(k)}$, of the systems in $\mathcal{E}_s$ that exhibit property $q^{(k)}$ can be counted and the fractions $N_s^{(k)}/N_s \equiv \{q^{(k)}|\mathcal{E}_s\}$ computed. When the number, N_s, is "large," an increase in the *size* of the tested sample will not materially change the distribution function that is obtained. Put in another way, if several samples of "sufficiently large" size are used - say $\mathcal{E}_{s1}$, $\mathcal{E}_{s2}$, etc., - distribution functions, $\{q|\mathcal{E}_{s1}\}$, $\{q|\mathcal{E}_{s2}\}$ etc., differ so little that they may be considered, for practical purposes, as equal (i.e., the differences constitute tolerable errors). Since we are now concerned with matters of *principle*, we are free to assume that the ensemble is so large that satisfactory distribution functions can be measured on samples (containing "large" numbers of systems) that are negligibly small by comparison with the parent ensemble. (This would be true, for example, if the sample contained $\sim 10^{-8}N$ systems, where N is the number of systems in $\mathcal{E}$.)

The number of systems affected by the measurement is so small compared with the total number in $\mathcal{E}$ (by assumption) that the ensemble directly after the test is still essentially the same as the ensemble just prior to the test (the interval Δt, between "just prior" and "just after" is assumed to be zero). The distribution on the sample, then, is a distribution function for the ensemble $\mathcal{E}$! Its meaning is best expressed in terms of probabilities. The value of the distribution function at $q^{(k)}$, i.e., $\{q^{(k)}|\mathcal{E}\}$, is the probability of finding (directly after the tests that measure $\{q^{(k)}|\mathcal{E}\}$) the property $q^{(k)}$, exhibited in a test for q, on a system selected at random from $\mathcal{E}$. Note that if $q^{(k)}$ is exhibited when an observable, q, is tested on a *single system*, no inference can be drawn about the property that will be exhibited in a second test of q immediately afterwards. (A test may even destroy the system tested; e.g., a test for photon energy may involve the absorption of the photon). Tests generally modify the systems tested in unpredictable ways. But a "distribution of q" in an ensemble may be measured without changing the "distribution of q." Hence *distribution functions* (but not system properties) may be considered to *have an "objective" meaning* (see subsection 7.1).

If two observables, q, s, are incompatible, it is impossible to test for both observables simultaneously. [Problem] However, it is easy to see that the *distributions* in the observables, q, s, can be measured simultaneously. Simply make the measurements for $\{q|\mathcal{E}\}$ and for $\{s|\mathcal{E}\}$ on different samples drawn from $\mathcal{E}$. Clearly it is possible, in principle, to measure the distributions of any set of observables on $\mathcal{E}$ at the same time. Note that "simultaneous distributions in incompatible observables" is meaningful, but that "distribution in two incompatible observables" is not.

If the systems in ensemble $\mathcal{E}$ *have* property $q^{(k)}$, i.e., $\mathcal{E}$ has been prepared by a $\mathcal{P}(q^{(k)})$ instrument, the distribution in q is given by

$$\{q^{(k)}|\mathcal{E}\} = 1 \quad \text{(all systems in } \mathcal{E} \text{ exhibit } q^{(k)} \text{ on test).}$$

$$\{q^{(i)}|\mathcal{E}\} = 0 \quad (i \neq k) \text{ (no system in } \mathcal{E} \text{ exhibits } q^{(i)} \text{ on test).} \tag{7.3}$$

Such a distribution is said to be "sharp" or "dispersion free" (see Fig. 7.9d). In other words, if the distribution of observable q is sharp in $\mathcal{E}$, then the systems in $\mathcal{E}$ *have* the property q. We need not speak separately of "properties of ensembles" and of "distributions in ensembles." A property corresponds simply to a special ("sharp") distribution. *All* the measurable features of an ensemble are known if the distributions in all the observables measurable on the systems in the ensemble are known. Therefore, two ensembles $\mathcal{E}_1$, $\mathcal{E}_2$, such that

$$\{q|\mathcal{E}_1\} = \{q|\mathcal{E}_2\}$$

for *all* observables q are said to be *equivalent*.

The *time dependence of the distribution of an observable* can be measured on an ensemble that changes its properties in time. Select distinct samples $\mathcal{E}'$, $\mathcal{E}''$, $\mathcal{E}'''$, etc., from the ensemble given at $t = t^0$; measure the distribution of q at t' on $\mathcal{E}'$, the distribution of q at t'' on $\mathcal{E}''$, etc. Notice that no system is subjected to more than one measurement ($\mathcal{E}'$, $\mathcal{E}''$, etc., contain no system in common). Consequently the distribution at t'' (say) is a characteristic of the ensemble initially prepared at $t = t^0$ after the time interval $t'' - t^0$. The measurements at t' do not influence the results of measurements at later times.[44] If the samples $\mathcal{E}'$, $\mathcal{E}''$, etc., together comprise only a small fraction of the total number of systems in the ensemble, the time dependence of the distribution may be measured without affecting the normal development of the ensemble.

The *distributions in microphysical observables* have many of the pleasing characteristics that are enjoyed by *classical observables*. Classical observables can be measured without changing the properties of a system, they are simultaneously measurable (compatible), and the motion of an observable can be measured without changing it in the course of measurement. None of these characteristics applies to microphysical observables; all apply to distributions of microphysical observables. These considerations suggest that distributions of observables will play, in microphysical theory, the important roles played by observables in classical theory.

7.5.3 *Average Value; Uncertainty*.

Just how much information we may need about a distribution of an observable depends on the character of the questions we seek to answer. Often it is sufficient to know merely that tests are effectively certain to discover positions in a definite *range* about some particular position. For a rough and often useful characterization of a distribution, the "distribution parameters," "average value" and "uncertainty" (defined below) are employed.

(1) *Average value (mean value, expectation value) of observable q in $\mathcal{E}$*. This is a physical magnitude denoted by $\{\overline{q}|\mathcal{E}\}$, or by the abbreviation $\overline{q}$ (when there is no ambiguity about the $\mathcal{E}$ to which it refers); it is defined by

$$\{\overline{q}|\mathcal{E}\} = \sum_k q^{(k)}\{q^{(k)}|\mathcal{E}\} \tag{7.4}$$

(the sum is over the properties in q). The $q^{(k)}$ in the product $q^{(k)}\{q^{(k)}|\mathcal{E}\}$ is a physical magnitude - a quantitative measure of the property, $q^{(k)}$, in some specific set of units. The average value is expressed in the units used in the measure of $q^{(k)}$ (Angstroms, electron volts, etc). Since $\{q^{(k)}|\mathcal{E}\} = N^{(k)}|N$ (see Eq. (7.1)) we have from Eq. (7.4)

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[44] Compare the measurement of the time dependence of a distribution with the measurement of the time dependence of an observable (i.e., the "motion of an observable") on a *system* (Subsection 7.4). In the latter case, succesive measurement of q at t''' does not reflect only the properties of the system prepared at t^0 but also the disturbances introduced by measurements at t', t''.

$$\{\bar{q}\,|\,\mathcal{E}\} = \frac{1}{N}\sum_k q^{(k)}N^{(k)} \qquad (7.4a)$$

(2) Uncertainty (dispersion, root mean square deviation) of q in $\mathcal{E}$. This is a physical magnitude with the units of $q^{(k)}$ (or $\bar{q}$) which is denoted by $\{\Delta q|\mathcal{E}\}$, or by the abbreviation Δq; it is defined by

$$\{\Delta q|\mathcal{E}\} = \left(\sum_k (q^{(k)} - \bar{q})^2 \, \{q^{(k)}|\mathcal{E}\}\right)^{\frac{1}{2}} \qquad (7.5)$$

Substitution of $N^{(k)}/N$ for $\{q^{(k)}|\mathcal{E}\}$ leads to

$$\{\Delta q|\mathcal{E}\} = \frac{1}{N^{\frac{1}{2}}}\left(\sum_k (q^{(k)} - \bar{q})^2 N^{(k)}\right)^{\frac{1}{2}} \qquad (7.5a)$$

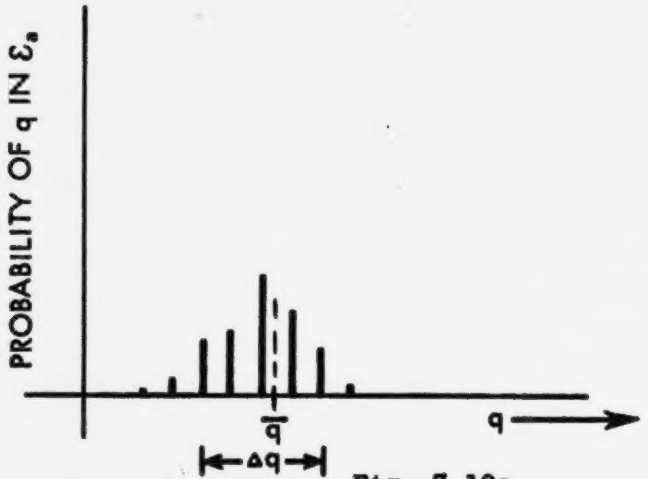

Fig. 7.10a

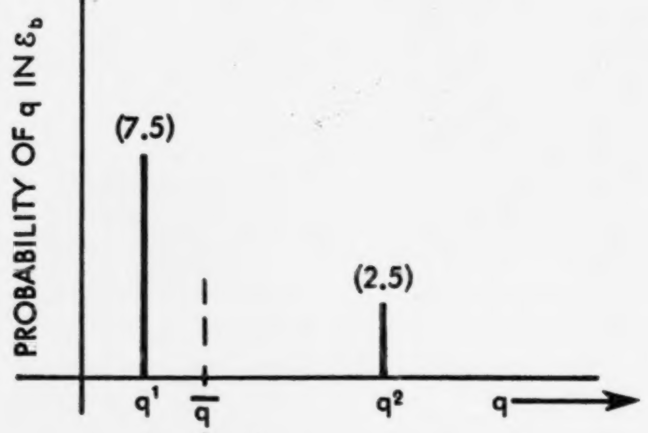

Fig. 7.10b

Fig. 7.10 Examples of the relation of the parameters "average value" $\bar{q}$ and "dispersion" Δq to distributions. In the distribution of q in $\mathcal{E}_a$ (Fig. 7.10a) the parameters provide a rough description of the distribution; in the case of $\mathcal{E}_b$, however, (Fig. 7.10b) the general character of the distribution is not described by $\bar{q}$, Δq.

For distributions of the sort shown in Fig. 7.10a, the probability of finding observable $\bar{q}$ (on test) with values in the range $q \pm \Delta q$ is of the order of one. The parameters, average value, and uncertainty provide a rough picture of the distribution. The average value and uncertainty provide a very poor characterization, however, of the distribution shown in Fig. 7.10b.

The distributions in all observables can be measured simultaneously; consequently the distribution parameters (average value, uncertainty) may also be measured simultaneously. Though x and p are not simultaneously measurable on an electron (with precision) (i.e., simultaneous x and p is not an electron property), the average values $\bar{x}$, $\bar{p}$ in any ensemble may be known simultaneously and precisely. In fact, the time dependences of the average values of all observables $\bar{x}(t)$, $\bar{p}(t)$, etc., in a given ensemble are measureable, objective properties of ensembles. These average values have, relative to ensembles of systems, many of the characteristics the functions x(t), p(t), etc., of classical physics, have relative to a system.

If the distribution of q in $\mathcal{E}$ is sharp, i.e., if $\mathcal{E}$ has one of the properties $q^{(k)}$ in q, it follows from Eqs. (7.4), (7.5), with the use of Eq. (7.3), that

$$\bar{q} = q_k, \quad \Delta q = 0 \quad (\mathcal{E} \text{ has property } q_k).$$

It is not difficult to show that if the uncertainty of q in an ensemble $\mathcal{E}$ is zero, $\mathcal{E}$ must have one of the properties in q. Thus an ensemble has a property contained in observable, q, if, and only if, $\Delta q = 0$.

Suppose that measurements of the properties in observable q are themselves inaccurate to within, roughly, a range δ. The results of measurements on an ensemble with property $q^{(k)}$ will be distributed, because of the inaccuracy of observation, within the interval between $q^k - \delta$ and $q^k + \delta$. On the other hand, if a second ensemble (of the sort shown in Fig. 7.10a) has an average value close to $q^{(k)}$ (say in the interval between $q^k - \delta/2$ and $q^k + \delta/2$) and has an uncertainty, Δq, less than δ, the distribution in the results of measurement will differ little from the distribution obtained for the ensemble with property $q^{(k)}$. Thus, by measurements of q with "error" δ, ensembles with a property (i.e., ensembles with $\Delta q = 0$), and ensembles with uncertainties $\Delta q < \delta$ are indistinguishable.

In our discussions of the Heisenberg principle, the symbols Δx, Δp_x were used to indicate roughly the ranges of position and momentum over which the position or the momentum of an electron, prepared by some given method, would be found on observation. These symbols correspond (again roughly) to the uncertainties in x and p_x (as defined by Eq. (7.5)) in an ensemble of electrons prepared by the given method. It will be shown later that the Heisenberg principle has an exact formulation in terms of uncertainties defined by Eq. (7.5).

7.5.4 Intermediate Measurements on Classical and Microphysical Ensembles.

The following considerations illustrate an error that is frequently made in the analysis of microphysical observations.

Suppose we want to find the distribution, $\{y|\mathcal{E}\}$, of observable y in an ensemble[45] $\mathcal{E}$ containing N systems and that the following information is given: (1) In $\mathcal{E}$ the distribution of q is specified by the numbers $f^{(k)} = \{q^{(k)}|\mathcal{E}\}$ and that only $f^{(1)}$ and $f^{(2)}$ differ from zero ($f^{(1)} + f^{(2)} = 1$). The numbers of systems $N^{(k)}$ found (on test of $\mathcal{E}$) with property $q^{(k)}$ are given by

$$N^{(k)} = N \cdot f^k,$$

(since $N^{(k)}/N = \{q^{(k)}|\mathcal{E}\} = f^{(k)}$).
(2) All ensembles with the property $q^{(k)}$ have the same distribution in y. Let $\mathcal{E}^{(k)}$ denote an ensemble with property $q^{(k)}$ containing $N^{(k)}$ systems, and let the known distribution functions $\{y|\mathcal{E}^{(k)}\}$ be denoted by $g^{(k)}(y)$. The number of systems in $\mathcal{E}^{(k)}$ that exhibit the property y' on test is $N^{(k)} \cdot \{y'|\mathcal{E}^{(k)}\} = N^{(k)}[g^{(k)}(y')]$. (This follows from the definition of a distribution: $N'/N^{(k)} = \{y'|\mathcal{E}^{(k)}\} \equiv g^{(k)}(y')$.)

It would appear that the desired distribution is easily calculated from the given facts by the following argument: Since the ensemble $\mathcal{E}$ consists of the subensembles $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ (with $N^{(1)}$, $N^{(2)}$ systems, respectively) which exhibit the properties $q^{(1)}$, $q^{(2)}$, on test, and the number of systems in $\mathcal{E}^{(k)}$ that exhibit property y' is $N^{(k)}[g^{(k)}(y')]$, the number of systems in $\mathcal{E}$ that will exhibit y' is

$$N' = N^{(1)}g^{(1)}(y') + N^{(2)}g^{(2)}(y').$$

Since N'/N is $\{y'|\mathcal{E}\}$ by definition, and $N^{(1)}/N = f^{(1)}$, $N^{(2)}/N = f^{(2)}$, we find, on dividing the equation above by N,

$$\begin{aligned}\{y'|\mathcal{E}\} &= g^{(1)}(y')\cdot f^{(1)} + g^{(2)}(y')f^{(2)} \\ &\equiv \{y'|\mathcal{E}^{(1)}\}\{q^{(1)}|\mathcal{E}\} \\ &\quad + \{y'|\mathcal{E}^{(2)}\}\{q^{(2)}|\mathcal{E}\}. \qquad (7.6)\\ &\qquad\qquad \text{(Classical)}\end{aligned}$$

This argument is correct for classical ensembles; it is false (in general) for microphysical ensembles. That Eq. (7.6) is false for microphysical ensembles can be proved by experiment (see example below). The error of the argument consists in the assumption that $\mathcal{E}$ is composed of the subensembles $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ where $\mathcal{E}^{(1)}$ has the property $q^{(1)}$ and $\mathcal{E}^{(2)}$ *has* the property $q^{(2)}$. However, all that measurement tells us is that *if the systems in the ensemble are tested for q*, the fraction $f^{(1)}$ *will exhibit* $q^{(1)}$ and the fraction $f^{(2)}$ *will exhibit* $q^{(2)}$. This does not imply, as has been emphasized frequently in this section, that the systems that exhibited $q^{(1)}$ (or $q^{(2)}$), on test, *had* the property $q^{(1)}$ (or $q^{(2)}$) before the test was made. Measurement, in general, changes the system measured.[46]

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[45] For simplicity, assume that the ensemble is time independent.

The argument leading to Eq. (7.6) (classical) was used in Section 4 in an effort to analyse the two-slit diffraction pattern (see Fig. 4.7). If the properties, α = "location behind slit a," β = "location behind slit b" (α and β are exclusive properties), are measured on the ensemble of electrons that pass the slits, equal numbers of electrons are found with the properties α and β, i.e., $f^{(\alpha)} = \frac{1}{2}$, $f^{(\beta)} = \frac{1}{2}$. If slit a is open and b is closed, all electrons that pass the slit system have the property α; if a is closed and b is open they all have property β. The distributions $\{y|\alpha\}$ or $\{y|\beta\}$, where y is a position component on the photographic plate, i.e., the one-slit patterns when the electrons *have* property α, or property β, may be observed. By the argument leading to Eq. (7.6) (classical) it is concluded that the two-slit distribution should be ($\mathcal{E}$ is the ensemble in two-slit case),

$$\{y|\mathcal{E}\} = \{y|\alpha\}\tfrac{1}{2} + \{y|\beta\}\tfrac{1}{2}$$

i.e., the two-slit pattern, according to (7.6) is a sum of single-slit patterns. Observation shows that this conclusion is false.

Clearly the distinction that must be made between "a system has a property $q^{(k)}$" and "a system exhibits, on test, the property $q^{(k)}$" is of great importance in microphysics. Because the two phrases have the same meaning in classical physics the difference between them is easily overlooked; the consequences are often catastrophic.

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[46] These objections do not apply to classical ensembles. In classical physics a measurement reveals the property possessed by a system.

8 DETERMINISM AND STATE; STATISTICAL DETERMINISM

> "He is no wise man that will quit a certainty for an uncertainty."
>
> Samuel Johnson

A clear appreciation of the significance of the classical concepts of determinism and state is essential for an understanding of the modifications required to adapt them to the needs of microphysics.

8.1 THE PRINCIPLE OF DETERMINISM AND THE CLASSICAL CONCEPT OF STATE.

Classical physics tacitly assumes that physical systems obey the principle of determinism; in effect, "like conditions produce like consequences and unlike consequences can only follow from unlike conditions." It is the principle of determinism that leads to the expectation that acorns will always grow into oaks and not, on occasion, into maple trees, or mulberry bushes, or tulips; and to infer that the oaks and tulips in the park did not all develop from acorns. In the technical language of physics the principle asserts that when two physical systems have identical properties[47] at some time, t_0, and are subject to the same external conditions, they will be alike in all properties at any later time, t_1; also, if two systems (under the given conditions) are different at t_1, they must have been different at t_0. Two ideal classical pendulums, for example, of the same length, and started at rest with the same deflections θ', from the equilibrium position, will have the same displacement at any later time. (If, however, the two pendulums have different deflection at t, they cannot both have been at rest with the same deflection at any previous time.) In classical physics, an isolated system, i.e., a system shielded from all variable external influences[48] always has a definite set of properties at an instant (even if they are unknown), and its develoment in time is completely determinate. An isolated system cannot choose different courses of development just as the acorn cannot choose to become a giant redwood. This feature of deterministic development enables astronomers to predict the occurrence of eclipses in the distant future from a knowledge of the present properties of the solar system.

Different systems of the same kind (same "fixed properties") may have different "motions." The word "state" is used to specify a particular motion from among all the possible motions of a system. For example, the two pendulums just referred to are in the same state of motion. A third pendulum started with the same displacement but with an initial velocity different from zero has a different state of motion. The word "state" is also used, not as a direct description of the whole motion, but to specify the particular set of dynamical properties possessed by a physical system at some instant. Thus the state of a pendulum is specified at an instant by giving the deflection and the velocity of its bob at that instant. By the principle of determinism the specification of the state of a system at some instant is equivalent to the specification of the motion of the system in time; two systems in the same state at t_0 must have identical motions, i.e., they have the same state of motion.[49]

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[47] Naturally, the identical properties must be internal to the two systems. Different external properties, such as location in different spatial regions, are assumed to be without effect. Examples of internal properties are: the distance between two mass points in the system, the relative velocity of two mass points, the internal energy (as against the kinetic energy of the center of the mass motion - an external property - which depends on the velocity relative to the frame of reference), etc.

[48] A simple example of an isolated system is a double star far removed from other stars; another example is our solar system.

Clearly two systems (of the same kind) are in the same state if they are alike in all internal dynamical properties. But how many is "all"? How, in practice, can we be certain whether two systems are alike in all their properties without a theory and without even a knowledge of the constitution of the systems? An example will clarify these questions and indicate how they may be answered. Suppose I am given a collection of simple harmonic oscillators (one-dimensional) all of which have a moving mass, m, and spring constant, k. Let's pretend I know nothing whatever about these systems to begin with, except that they are all of the same kind. Suppose that after many investigations I have discovered various observables - in particular, an energy observable, a location observable (the distance of the oscillating mass relative to the equilibrium position of the oscillator), and a momentum observable (the momentum of the moving mass in a frame of reference in which the equilibrium position of the oscillator is at rest). My object is to discover what properties my systems need have in common at some instant in order to assert that they are "in the same state."

The investigation may be started in many different ways. I begin, say, with the energy observable. Since systems in the same state must be alike in all properties, they must certainly have the same energy. Therefore I select, by an energy measurement, a large number of systems with the same energy property, E′, and observe whether a second property - say location - is the same for all the selected systems. Tests show that the systems that are alike (uniform, homogeneous) in the energy property are variable in location. Systems with the same energy E′, then, are not in the same state. In order to reduce the observed variability, I select a set of systems with the same energy, E′, and the same location, x′, and observe whether all these systems have the same momentum.[50] Only two momentum values are found and these differ only in sign. In other words, when systems have the same energy and location properties, they also have the same "magnitude-of-momentum" property; the magnitude-of-momentum is uniquely determined by the energy and location properties already measured.[51]

My general strategy is easily summarized (Fig. 8.1a). Whenever variability in an observable q is discovered (i.e., measurements reveal the properties q′, q″, etc., belonging to the observable q) in a set of systems with certain common properties r′, s′, I seek a new set in which the variability in q is also eliminated. Since, for example, oscillators with energy, E′, location, x′, vary in the sign (±) of the momentum (let κ designate the observable "sign of momentum") I proceed by selecting a set of systems

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[49] A motion is described completely by the set of functions q(t), r(t), etc., which specify the dependence of all system-observables on time. The equivalence of "state at an instant" and "state of motion" means that the system properties q(t′), r(t′), etc., at instant t′ uniquely determine the functions q(t), r(t), etc.

[50] It must be imagined that the momentum measurement is made immediately after a system is prepared with the chosen energy, E′, and location, x′.

[51] All these results are obvious, of course, from the theory of the oscillator. The energy, E, is related to the displacement, x, and momentum, p, by

$$E = p^2/2m + \tfrac{1}{2}kx^2$$

Hence, $x = \pm\sqrt{2E/k - p^2/mk}$, $p = \pm\sqrt{2mE - mkx^2}$

Thus x is not determined by E alone, but p is determined to within a sign by E and x.

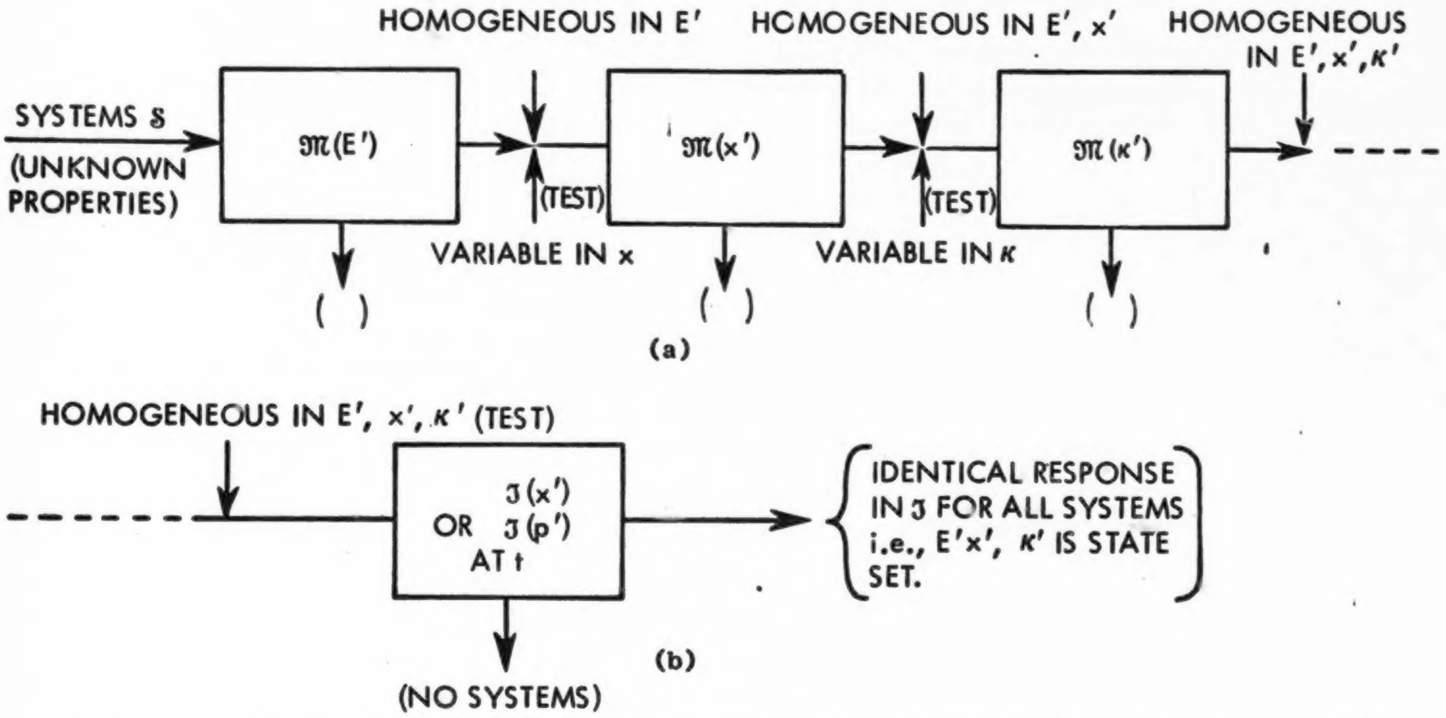

Fig. 8.1 Diagram of procedure for discovering state sets of properties. (Classical)

homogeneous in E', x', and κ'.[52]

I can no longer go on quite as before since I only know how to measure energy, location, and momentum. Therefore I test whether systems with the properties E', x', κ' at some instant, t_0, are in the same state by observing whether the systems vary in some property - say location - at a later time, t_1 (i.e., I investigate whether the systems are in the same state by testing whether the given properties uniquely determine the motion). I find that all systems have the same location property at t_1. Similar tests made at various times of both location and momentum properties indicate that systems with properties E', x', κ' at t_0 are the same in any property measured at any time (see Fig. 8.1b). I conclude, therefore, that systems with the same energy, location, and sign of momentum, at some time, are in the same state; i.e., all properties at t_0 are the same in a set of systems if each system has the properties E', x', and κ'. Similar investigations would show that a set of systems homogeneous in x' (location) and p' (momentum) are in the same state and that, if $\rho' = \kappa'[2mE' - m\kappa(x')^2]^{\frac{1}{2}}$ (see footnote 56), systems with the properties x', p', and systems with properties E', x', κ', are in the same state.

Suppose that in the initial investigations of the system I had failed to discover a momentum property. After selecting a set of systems homogeneous in energy and position at t_0, I could test whether they are in the same state by making location measurements at later times. I would discover that location at t_1 is not the same for all systems. Using the principle of determinism I would then infer that the systems differed in some property at t_0. Note that the principle of determinism directs a search for additional properties. Unless a group of properties are found such that systems homogeneous in these properties

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[52] Note that κ is a nonquantitative observable with only two properties (designated by the symbols + and −). The symbols E', x' designate specific energy, location properties; similarly κ', which may be either + or −, indicates a particular property of the sign of momentum observable. E' or x' may be specified by numbers related to appropriate units; e.g., E' = 3.0 eV, x' = −.5 Å. Obviously κ' has no corresponding numerical expression.

are the same in all known properties at all times, the principle of determinism asserts that more properties are discoverable.

Two useful technical definitions may be understood by reference to the foregoing example. A *set* of properties, u', (such as (x', p') or (E', x', κ')) is called a *set of state properties* ("state set"), for systems of kind S, if, whenever different S systems have the same properties, u',[53] at some instant, all properties of the S systems are the same at all times; i.e., different systems with the same state properties, u', are in the same state. A set of properties, v', is called an *independent* set of state properties if: (a) v' is a set of state properties, and (b) no property in the set v' is uniquely determined by the remaining properties in the set.[54]

Suppose someone came to you and said, "You're a student of physics. Perhaps you can help me. In the next room I have a simple harmonic oscillator with a mass, m, and spring constant, k. Exactly how far will the mass be from its equilibrium position at two o'clock this afternoon?" You could of course give no answer. You might explain that physics *never* gives answers to questions such as the one proposed; a knowledge of system constitution is not sufficient to permit a prediction of a dynamical property of the system. "What else," he may then ask, "must I tell you to enable you to answer my question?" A sufficient answer is "any set of state properties of the oscillator at some definite time."

This answer illustrates the central significance of the state concept. The *determinate* connections among phenomena that are discoverable by experiment or predictable by theory are: (1) the connections between a set of independent state properties at some time and the properties *dependent* upon the state set at the same time (e.g., the relation between x, p for the oscillator and the energy of the oscillator; see footnote 54); (2) *the connection between state properties at some time and any property at another time* (e.g., given x_0 and p_0 for the oscillator at t_0 there is a unique x_1 at t_1 which may be known experimentally or calculated from the laws of motion). The exact predictions of physics are essentially of this form:[55] *If* the set of independent *state* properties of system S at t_0 is u', *then*, at t_1, property r' will be found. In short, precise predictions of all system properties are possible only if a *set of state properties* (i.e., the *state of the system*) is known.

8.2 THE PRINCIPLE OF STATISTICAL DETERMINISM

It's easy to see that the classical concept of state can't play a role in microphysics. The concept is wholly dependent on the principle of determinism, and microphysical phenomena simply aren't governed by this prin-

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[53] The symbols u', v', w' will be used to denote *sets* of properties.

[54] For oscillators, the properties energy, location, and sign of momentum form an independent set of state properties: no two of these properties uniquely determines the third. Location and momentum is a second independent set of state properties. Energy and location properties are independent (the energy property does not uniquely determine the location property of an oscillator) but this pair is not a set of state properties. Energy, location, and momentum magnitude are *dependent* properties since momentum magnitude is uniquely determined by the energy and location. A set of state properties for the simple harmonic oscillator is "energy, location, momentum magnitude, momentum sign"; these properties are not independent since momentum magnitude is uniquely determined by the energy and momentum. Note that if any property is omitted from an independent set the remaining group is *not* a set of state properties. (Why not?)

[55] Note, however, that a complete set of state properties is not always required for prediction. For example, the properties E', x', of an oscillator - which do not form a set of state properties - uniquely determine the momentum magnitude, p'.

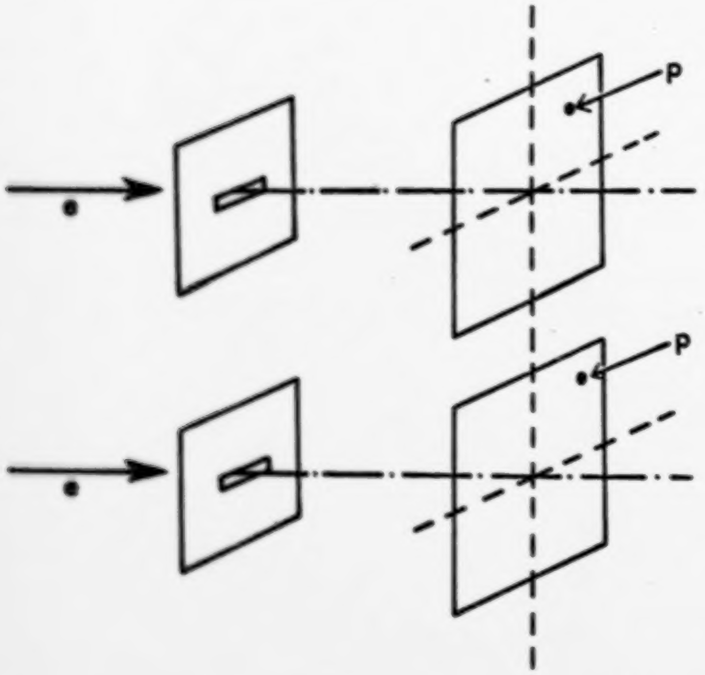

Fig. 8.2 An impossible experiment. Electrons behind the entrance slit cannot be so prepared as to determine their positions at the detector.

ciple.[56] A number of examples of the indeterminate behavior of microphysical entities were cited in previous sections. The location at which an electron in a diffraction experiment will arrive at a detecting screen cannot be controlled. If we set up two single-slit diffraction experiments, it is impossible to prepare an electron in each apparatus so as to be certain that both will arrive at their respective detecting screens at the same relative position (Fig. 8.3). It is impossible, similarly, to prepare two neutrons so as to be certain that both will turn into protons at the same time. In other words, we cannot prepare two electrons or two neutrons in the same state (in the classical sense).

From the existence of incompatibility, one may show that classical determinism cannot hold in microphysics; if classical determinism were true, no system properties could be incompatible. Suppose that an ensemble of systems is prepared so that its systems are in the same classical state, and that (for example) position x and momentum p_x are observables for the system. Let x be tested on half the ensemble, p_x on the other half. Since all systems are, by assumption, in the same state, all systems in the half tested for x must exhibit the same position - say x′ - and all systems in the other half must exhibit the same momentum - say p_x'. (Systems are in the same classical state only if responses to identical tests are identical.) But these results imply that both x and p_x have "sharp" distributions in the original ensemble (Subsection 7.5), i.e., the ensemble was prepared with both an exact position and an exact momentum property. The existence of such a preparation implies further that the measured properties are compatible. Since the same argument may be applied to any set of observables, it is clear that if it is possible to prepare systems in classical states, then all the properties of the systems are compatible.

Recall that to predict the result of any test on a system, a knowledge of its state properties at an instant is necessary. Since in microphysics the classical state concept fails, it is not possible to prepare a system so as to be certain of the outcome of any test that might be made upon it. This is what is meant by the "indeterminateness" of microphysical systems.

The behavior of microphysical systems, though clearly indeterminate in some features, is far from chaotic. In complete chaos, experiment could discover no regular relations among phenomena, and all theory could tell us is "Your guess is as good as mine." Many regular and determinate relations have been described in the preceding sections. We have noted that definite diffraction patterns can be repeatably

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[56] If the principle of determinism is taken as a necessary truth or a metaphysical necessity, one is driven to the assumption of "hidden variables." Reasons for rejecting this assupmtion were considered in Section 6.

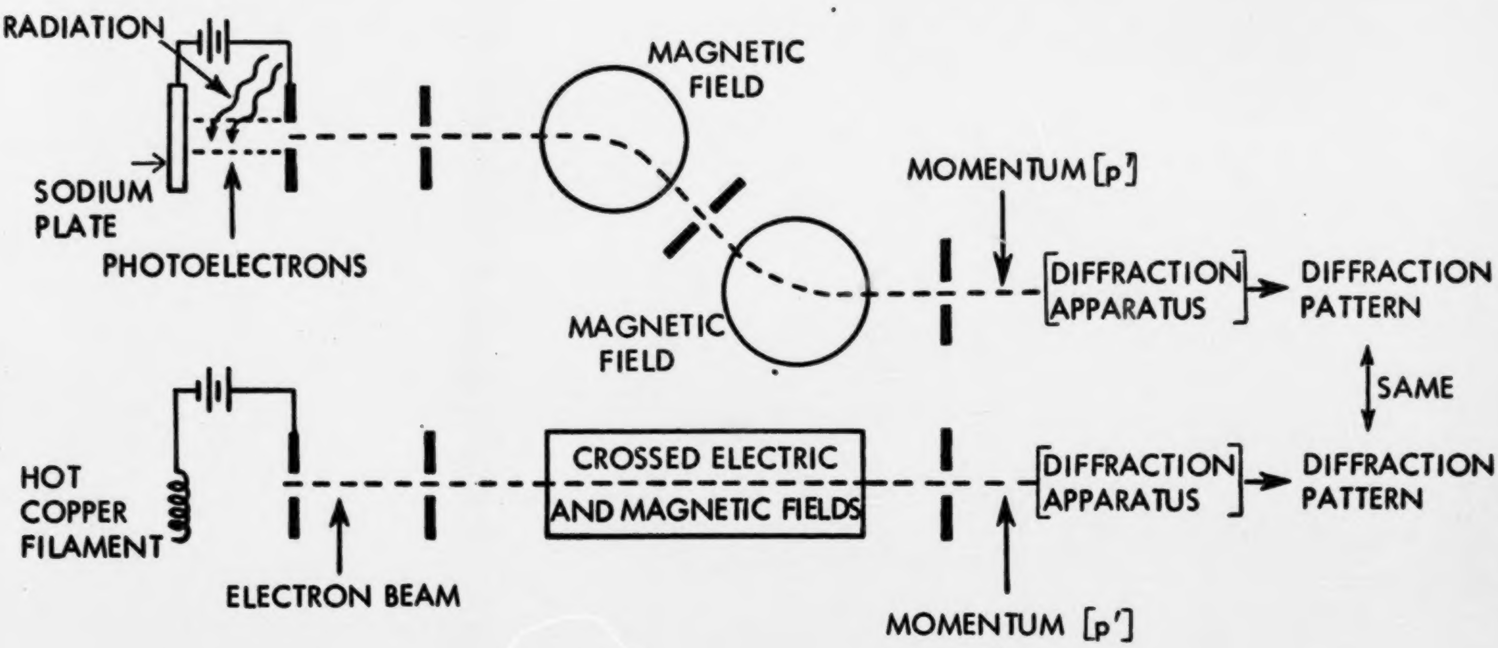

Fig. 8.3 Example of equivalent state ensembles |p′] with different prior histories.

produced; radioactive substances exhibit definite half-lives. Many other regularities are obvious: the spectrum of hydrogen observed today contains the same lines Balmer measured many years ago; a gas of hydrogen molecules has well-defined, stable properties, and so on. Obviously a great deal of regularity may be found in microphysical behavior despite the appearance of indeterminacy; theory has much to "explain."

In all physics (macro-and micro-), observable regularities are of the form: "If the object (entity, ensemble) of interest has certain properties, then some other properties of the object will be observed"; or, put in a different way, "If physical apparatus is arranged in some definite way, then certain other manifestations will be observed when tests are made." If, for example, a classical pendulum is started from rest with a deflection θ', then we shall find that deflections larger than θ' never occur: If the electron momentum of the incident beam in a diffraction experiment is sufficiently well defined in magnitude and direction, then a characteristic diffraction pattern is produced. In short all observable regularities have an "if _____, then _____" structure.

It has been shown that in classical physics the blank after "if" must (in general) be filled in with a set of state properties; if a certain state is prepared, then some other property will be found. Is it possible to give a similar characterization for the contents of the blank after "if" for microphysical systems? The answer is yes; the necessary content of the blank after "if" may be discovered by examining more closely the implications of the existence of determinate relations in microphysics.

In classical physics, the concept of state depends essentially on the principle of determinism; because this principle is invalid in microphysics, so, too, is the concept. Now the principle of determinism expresses, in effect, the existence of precise, determinate relations among observations. But such relations also occur in microphysics. This suggests that it should be possible to formulate a principle for microphysical processes which plays the same role relative to the regularities of microphysics that the classical principle of determinism plays in classical physics.

Although we can't predict where a single electron will appear on a detecting screen in a diffraction experiment, we can certainly predict the diffraction patterns that are produced in experiments such as the one described in Fig. 4.1. Two diffraction experiments with different apparatus (but both with the arrangement shown in Fig. 4.1), and different ensembles of electrons (but with the same large total number) produce indistinguishable patterns. Similarly, the exact time at which a radioactive nucleus will experience a transformation cannot be controlled or predicted, but the fraction of the initial number of nuclei which remain after a lapse of time, t, is the same for different ensembles of radioactive nuclei of the same kind. These observations are typical; along with the indeterminate behavior of single systems, determinate (regular, repeatable) behavior of ensembles of these systems is observed. This suggests that a principle of determinism holds in microphysics, not for single systems, but for ensembles of systems.

Diffraction experiments reveal the determinate relation: if an ensemble of electrons is formed (e.g., by the use of the source and accelerating plates as in Fig. 4.1) with a well-defined momentum, then the distribution in position of the electrons over the plane of the detecting screen forms a definite and reproducible pattern. Similarly, from observations on neutron decay we can conclude: If an ensemble of free neutrons is prepared, the fraction of the initial number of neutrons that remain after interval, t, is a definite and reproducible exponential function of t. These regular relations have the general structure: If certain conditions are true of an ensemble of microphysical systems, then the distribution function of some observable has a definite form.

It is now possible to discern the content of the microphysical principle for which we are searching. To distinguish it from its classical counterpart and to suggest its general character it is called the Principle of Statistical Determinism. The classical and microphysical principles may be given parallel formulations.

Classical Principle of Determinism:

It is always possible to find a set of properties - generally many different sets - such that whenever two systems are alike (at t_0) in all the properties of the set, they are also alike in any property (at t_0 or any time t_1). If two systems differ in any property at t_1, then they must have been different in some property (or properties) at a previous time t_0.[57]

Microphysical Principle of Statistical Determinism:

It is always possible to find a set of compatible properties - generally many different sets - such that whenever two ensembles of systems are alike (at t_0) in all the properties of the compatible set, then the distribution in any observable is the same in both ensembles (at t_0 or any later time t_1); if two ensembles exhibit different distributions in any observable at t_1, then the ensembles must have been different in some property at a previous time t_0.

A very long and possibly dangerous inductive leap is necessary to get from the few examples of determinate statistical relations mentioned in previous paragraphs to the grand generalities of the principle of statistical determinism. A theory must, of course, presuppose a deterministic principle of some kind, and the one formulated above is at least consistent with the microphysical observations we have analyzed. The new principle may be considered as a generalization of the classical principle of complete determinism; if we put in place of "the distribution of any observable is the same...," the words "the distribution of any observable is *sharp*, and is the same...," the resulting statement has exactly the meaning of classical determinism. Thus, in accepting the new principle, the possibility of complete determinism is not ruled out since no restriction is imposed on the character of the distributions in the ensemble to which the microphysical principle refers.

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[57] For brevity a number of obvious but necessary stipulations have been omitted. The systems are of the same kind (same fixed properties); between t_0 and t_1 (the interval $t_1 - t_0$ is arbitrary), such external influences as may affect the system are presumed to be identical for both systems (or all the systems in the two ensembles). Observation is not permitted on the ensembles in the interval between t_0 and t_1 since observations produce, in general, large and uncertain system changes. If the same observations are made on both ensembles (in the t_0, t_1 interval), the distributions observed at t_1 are again the same for both ensembles, but, in general, the distributions so obtained are different from those found if the intermediate observation is omitted (see the discussion of the intermediate measurement in Subsection 7.5.4).

The crucial question is: Can laboratory observations be fitted naturally, i.e., without the constant addition of ad hoc assumptions, into a theoretical structure based on statistical determinism? One can only say that the theory of quantum mechanics, which incorporates the principle of statistical determinism, has been highly successful in the realm of microphysics; as yet there is no evidence that suggests the invalidity of the principle. (But remember that after two centuries of successful application of classical determinism, experience in microphysics forced its abandonment.)

8.3 THE MICROPHYSICAL CONCEPT OF STATE

It is readily seen that the state concept (classical) defined in Subsection 8.1 is closely related to the classical principle of determinism (D). The set of properties referred to in D is a *state set* of properties; according to D, whenever systems are alike, at some instant, in the properties of a state set, they are alike in all properties at any time, i.e., the systems are in the same state (or state of motion).

It is natural to define the state concept for microphysics so that it relates to the principle of statistical determinism, S.D., exactly as the classical concept relates to D. Any set of properties, then, of the kind referred to in S.D., will be called a "state set" of properties for the systems composing the ensemble.[58] According to the S.D. principle, all *ensembles* that have the same state sets at some instant are *equivalent* at any later time (the distributions in these ensembles are alike in all observables). Just as the "motion of a system" means simply the time dependence of its observables, the "motion of an ensemble" means the time dependence of the *distributions* of its observables. Consequently the principle of statistical determinism may be expressed: ensembles with the same state sets of properties have the same *motion*.

An ensemble with a state set of properties is called a state ensemble. Different ensembles with the same state sets are said to be in the same state;[59] therefore ensembles in the same state are equivalent. The term "state," clearly, has been defined for *ensembles* and not for single systems. When, however, an ensemble is *known* to be in some particular state, no complications arise if each system in the ensemble is said to be in that state. Note that for such systems a *known* state set of properties is *possessed* by the system.

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[58] The number of properties required to make up a state set for systems of some kind depends on what properties are chosen. For an ensemble of hydrogen atoms, the property "energy = −13.6 eV" is, by itself, a state set (note that a state set may be composed of a single property); a particular set of components of the vector, $\overline{r}$, from the electron to the proton, x', y', z', is also a state set (subject to the limitations of footnote 61). However, no state set x', y', z' defines the same state as the property "$E' = -13.6$ eV"; the two state sets are incompatible.

Clearly the properties contained in state sets are compatible. If observables q, r, ..., are such that *all* the compatible sets of properties q′, r′, ..., formed from these observables are state sets, the observables q, r, ..., are said to constitute a "complete set of compatible observables." The sets of observables (for hydrogen atoms): "the components x, y, z of the vector from proton to electron" and "energy, angular momentum magnitude, z component of angular momentum" are complete (subject to the conditions referred to above).

It is useful to have special notations for state sets and state ensembles. That an ensemble $\mathcal{E}$ is a state ensemble will be indicated by writing $|\mathcal{E}]$; similarly, if a set of properties u′ is a state set, we denote the set by the symbol $[u']$. Thus after it is established that (say) a pair of properties q′, r′ constitute a state set, we shall write $[q', r']$ instead of q′, r′. All ensembles with the same state set of properties $[u']$ are equivalent (they differ, as ensembles, only in the numbers of systems they contain); consequently, they may be denoted by the same symbol. We shall use $|u']$ to denote any ensemble with the state set $[u']$. (Distinguish carefully between $[u']$, a *state set* of properties, and $|u']$, a *state ensemble.*) The distribution function of an observable, s, in a state ensemble, $|u']$[60] will be indicated by the symbol $\{s|u']$. This distribution, it will be recalled, is a function defined over the domain of the properties in the observable s. The symbols introduced above and a few others, together with their meanings, are collected in Table 8.1

(a) A "state set of properties u′" (also "state properties")	$[u']$
(b) A "state ensemble $\mathcal{E}$"	$|\mathcal{E}]$
(c) An "ensemble with the state set $[u']$"	$|u']$
(d) "Distribution of observable s in $|u']$"	$\{s \mid u']$
(e) "Probability of property s′ in $|u']$"	$\{s' \mid u']$
(f) "Probability of *state set* $[v']$ in $|u']$"	$[v' \mid u']$
(g) "Ensemble prepared initially in state $|u']$ *after interval t*"	$|u'; t]$

Table 8.1

With the help of the definitions of Table 8.1, the principle of statistical determinism may be expressed as follows: "The time dependent distribution $\{s|u';t]$ (when s is any observable) is uniquely determined by the state set $[u']$ prepared at t = 0"; or, more simply, as: the motion of an ensemble is uniquely determined by the initial state set $[u']$. It follows, by induction from many experiments or by suitable theory, that all statistical properties testable on a state ensemble are predictable from a knowledge of the state properties with which the ensemble is prepared. Note in particular that *the history of an ensemble* (or of its component systems), *before its preparation as a state ensemble*, has no influence on its statistical properties *after* the preparation; by the S.D. principle, whatever their prior histories, all ensembles, with the same state properties are necessarily equivalent (see Fig. 8.3 for an example). From these considerations it is obvious that the blank after "if" in the "if_____, then_____" statements of microphysics must contain a state set of properties and the blank after "then" may contain any testable statistical property.

It is known, for example, that the properties of hydrogen atoms, "energy E′, angular momentum magnitude ℓ′, z component of angular momentum

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[59] In the literature of quantum mechanics, the term "pure state" is used instead of the unmodified word "state." The adjective, "pure," adds no content to the term; it is employed for emphasis as is the word "honest" in "honest truth." Generally phrases such as "honest truth" are used when the "truth" is doubtful or when one expects doubt from a listener. I imagine that the word "pure" in the term "pure state" implies: the ensemble *really is in a state* even if (on classical principles) it doesn't much look like one.

[60] Often "state, $|u']$" rather than "state ensemble, $|u']$" is used; also (somewhat inconsistently) the phrase, "ensemble in state $|u']$."

m_z'," constitute a set of state properties.[61] These properties are "constants of motion," i.e., they do not change with time; consequently a state ensemble with properties $\lfloor E', \ell', m_z' \rfloor$ remains a state ensemble with the same properties at any later time. Thus the distributions in all observables are independent of time and no reference need be made to the time at which a distribution is measured. (Such ensembles, or states, are called "stationary.") In the state ensemble (of hydrogen atoms) $|E', \ell', m_z'|$, the distribution of the observable r (r = electron proton distance, say) is given by the function _____; the distribution in the observable m_x (the x component of angular momentum) is given by the function _____; etc. The blanks refer to specific functions (tables, graphs). In the case of the diffraction experiments of Section 4, the incident ensemble is characterized by the properties, "$p_x' = 0$, $p_y' = 0$, $p_z' = p_0$;" these constitute a state set of properties for electrons. This state set determines a unique distribution function $\{y|p']$ of the observable y (distance from axis of apparatus transverse to the beam and perpendicular to the slits - see Fig. 4.7), which describes the observed diffraction pattern. Our last example illustrates time dependence. The property n' = "number of neutrons at rest" is a state set of properties. If a state ensemble $|n']$ is prepared at $t = 0$, then the probability of observing $\lfloor n'' \rfloor$ at time t is $\lfloor n'|n'; t \rfloor$ (see Table 8.1, entries f and g); this function is found by observation to be $n'[\exp(-\lambda t)]$ where λ is a constant.

An ensemble that is not a state ensemble (i.e., not all systems in the ensemble are the same in some state set of properties) is called a mixture. This concept may be illustrated by the following simple example. Let $|u']$ and $|u'']$ be different state ensembles containing N' and N'' systems, respectively. The collection of $N = N' + N''$ systems consisting of the N' systems in state $|u']$ and the N'' systems in state $|u'']$ may be considered as an ensemble $\mathfrak{M}$; this ensemble is a mixture. It is said to be a mixture of the states $|u']$, $|u'']$ with the weights N'/N, N''/N.[62]

To examine another aspect of the microphysical concept of state, it is necessary to begin with a definition of an "indecomposable ensemble." Let $\mathcal{E}$ be an ensemble consisting of the systems $\mathcal{S}_1, \mathcal{S}_2, \ldots \mathcal{S}_N$. $\mathcal{E}$ may be divided into subensembles (two or more), $\mathcal{E}_1, \mathcal{E}_2, \ldots$, in many different ways and by many different procedures. For example: (1) divide $\mathcal{E}$ by simply considering that the odd-numbered systems $\mathcal{S}_1, \mathcal{S}_3, \mathcal{S}_5$, etc., constitute $\mathcal{E}_1$ and the remaining systems constitute $\mathcal{E}_2$; this is simply a mental decision which has no influence on the systems; (2) subject $\mathcal{E}$ to a preparatory measurement of observable q and form the subensembles $\mathcal{E}_1$, of the systems with property $q^{(1)}$ (after the measurement), $\mathcal{E}_2$, of the systems with property $q^{(2)}$, etc.; in general, this measurement process changes the properties of the systems in $\mathcal{E}$. Let $\mathcal{E}_a$ be the ensemble consisting of the systems in all the subensembles, i.e., of all N systems, after the subdivision. Some subdivisions, e.g., the subdivision of our first example, are such that $\mathcal{E}_a$ and $\mathcal{E}$ are equivalent (we shall denote this equivalence by $\mathcal{E}_a \sim \mathcal{E}$); other subdivisions may produce an $\mathcal{E}_a$ that is not equivalent to $\mathcal{E}$ ($\mathcal{E}_a \not\sim \mathcal{E}$)[63]. An ensemble is called "indecomposable" if, whenever a subdivision is such that $\mathcal{E}_a \sim \mathcal{E}$, the subensembles formed in the division are all equivalent to $\mathcal{E}$ (and therefore to each other); i.e., if whenever $\mathcal{E}_a \sim \mathcal{E}$ also $\mathcal{E}_1 \sim \mathcal{E}_2 \sim \mathcal{E}_3 \ldots \sim \mathcal{E}$, then $\mathcal{E}$ is indecomposable. A decomposable ensemble is one that can be subdivided "without change" (i.e., $\mathcal{E}_a \sim \mathcal{E}$) into *inequivalent* subensembles.

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[61] Spin properties and the interaction of the atom with the electromagnetic field are neglected in these statements. Also only the "internal" properties of the hydrogen atom are considered.

[62] Examples of mixtures are considered in Chapter 2.

[63] Individual systems may be changed in the subdivision; $\mathcal{E}$ and $\mathcal{E}_a$ are equivalent if the statistical properties of the ensemble are not changed.

Consider an ensemble of classical systems. If the responses to tests (on the systems of the ensemble) of an observable - say q - are variable, then the systems in the ensemble are not all in the same state. Also the variability indicates that the ensemble is *decomposable*. Measurements on classical systems may be made without change of the system properties; thus the ensemble may be decomposed "without change" into subensembles in each of which the systems have a particular property in q. Obviously the subensembles are inequivalent - their distributions in q are different. It follows that an ensemble of classical systems is indecomposable if, and only if, all the systems in the ensemble are identical, i.e., all are in the same state. (indecomposability, it may be noted, constitutes a criterion for judging whether the systems in the ensemble are the "same," "identical.") Clearly the state ensembles in classical physics are the indecomposable ensembles.

This suggests a new definition of "state," applicable both to classical physics and microphysics. Systems in indecomposable ensembles are, by this new definition, in the same state. If an ensemble is indecomposable, then it is a state ensemble; if decomposable, it is a mixture. In the quantum mechanics, as we shall show later, the two definitions of state ensembles - in terms of a state set of properties and in terms of indecomposability - are equivalent.

The difference between the state concepts in classical physics and in microphysics may be described as follows. In classical physics, ensembles are decomposable unless tests of any observable produce the same response on each system in the ensemble; thus, in indecomposable classical ensembles all distributions are sharp. In microphysics the distributions in indecomposable ensembles are *not* all sharp.

In principle, state sets of properties for systems of some kind may be discovered experimentally by procedures similar to those used to find classical state sets (Subsection 8.1; see Fig. 8.1). Naturally - as in the classical case - it is necessary to know how to measure a variety of observables on the systems under consideration (Section 7). Suppose that instruments that measure observables ℓ, m, q, r, s ... are available, i.e., we have instruments, $\mathfrak{M}(\ell)$, $\mathfrak{M}(m)$, etc. We choose an observable, say q, and prepare a collection of ensembles with the property q′. The different ensembles are tested for equivalence by measuring the distributions in all other known observables and the time dependence of these distributions. If the distributions in all ensembles prepared with property q′ are the same at all times, we conclude that the property q′ is a state set and that an ensemble with property q′ is in state |q′]. If, however, in different ensembles with property q′, the distributions of some observable s are not all identical, q′ is not a state set of properties. The principle of statistical determinism then assures us that there exist properties (or at least one property) compatible with and independent of q′.[64] Let r′ be such a property. Ensembles with both q′ and r′ properties are prepared and tested for equivalence. If ensembles with properties q′, r′ are equivalent whenever, wherever, and however they are found, then [q′, r′] is a state set and an ensemble with this pair of properties is in the state |q′, r′]. If the properties q′, r′ do not constitute a state set, i.e., if ensembles with this pair of properties are not *necessarily* equivalent, then further independent and compatible

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[64] The terms, "dependent," "independent," as applied to compatible microphysical properties have the same meanings as in classical physics. However, for classical properties there is no need to add the adjective, "compatible."

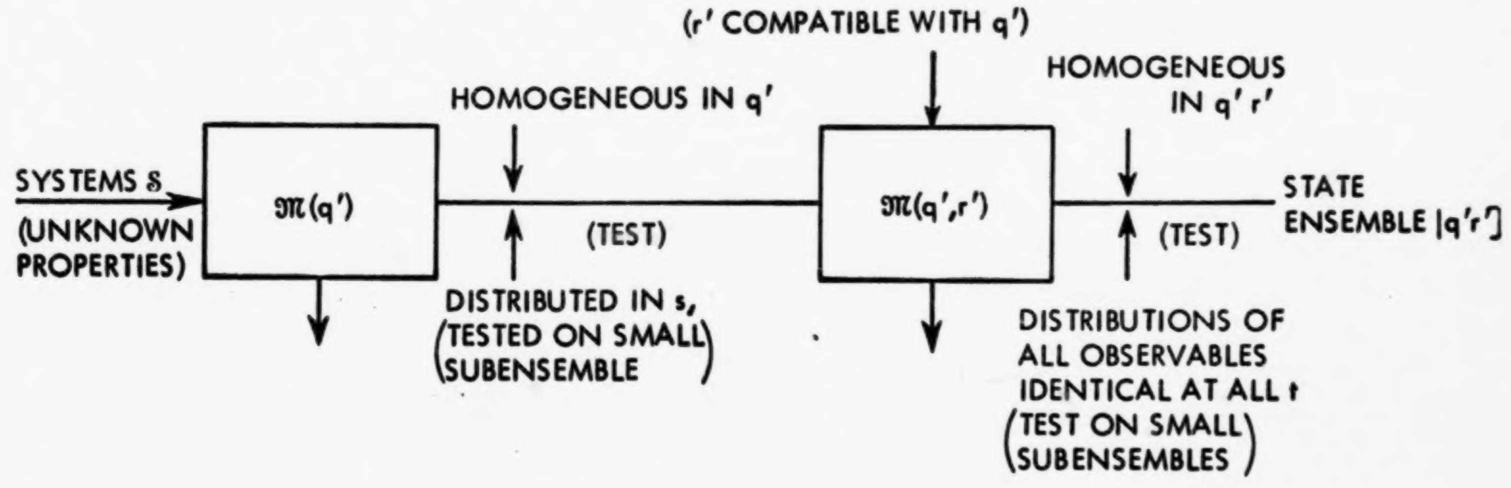

Fig. 8.4 Diagram of procedure for discovering state sets of properties. (Microphysical). Compare with Fig. 8.1.

properties exist. The general form of the process is clear. Until equivalent ensembles are achieved, properties independent of and compatible with those already employed are added. The principle of statistical determinism assures us that this procedure, if sufficiently extended, will always lead to a state set of properties. (The procedure outlined above is summarized in Fig. 8.4.)

The microphysical concept of state is subtle and easily misinterpreted. Clear differences between the classical and microphysical state concepts are easily distinguished. Since, in classical physics, system properties have *objective* meaning, *any system is always in a state* whether or not its properties are known. It is *not* true, however, that any ensemble of microphysical systems is a state ensemble. Moreover, the statement, "A single microphysical system, S, is in some state although its state properties are unknown" is *without testable meaning*, or, more bluntly, is meaningless.[65] [Problem]

It may be shown that any mixture ensemble (i.e., not a state ensemble) is *equivalent* to an ensemble obtained by putting together, with appropriate weights, a number of different state subensembles. It is possible, however, to produce equivalent mixtures (ensembles with the same statistical properties) by composing quite different state subensembles; consequently it cannot be said that energy mixture is formed of some unique combination of states (this may be said of "classical mixtures," i.e., ensembles of classical systems in different "objective" states). Classically, if two systems are alike in all their properties they are in the same state. However, if two microphysical ensembles are equivalent they certainly need not be in the same state. *Equivalent mixtures* are easily produced. [Problem] If two ensembles are in the same state they are necessarily equivalent; but the equivalence of two ensembles does not imply that they are state ensembles, or even that they have been formed by the same mixtures of state ensembles.[66]

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[65] The sentence quoted has exactly as much meaning as: "There are numerous physical bodies all about us that experience no interactions whatever with the entities of which we are capable of having any knowledge."

[66] Examples illustrating many of the remarks of this paragraph will be given in Chapter 2.

9 SUMMARY AND COMMENT

> "And, no matter what sort of Hell hath popped,
> Let not the constant h be dropped."
>
> —Leonard Bacon

9.1 EPITOME

A first reading of a set of new ideas often results in confusion, especially if the organization of that set is complex. Each new concept reruires detailed study; the successive focusing of attention on small parts makes it difficult to perceive the design of the whole. Yet, for the work to follow, an approximation to an understanding of the whole is what is required. A summary may help clarify the structure and intent of the work of this chapter.

Since classical theory fails so completely in the realm of microphysics, it is obvious that the conceptual tools it provides for the study of microphysical entities cannot be trusted. However, before the inadequacy of classical conceptions became evident, many microphysical observations were interpreted classically. It is important, therefore, to differentiate clearly between classical interpretation and the brute facts of observation (Section 2). A reexamination of all classical interpretations of experiments in microphysics obviously could not be undertaken. Fortunately, a number of the properties of electrons and photons are obtained by methods that are hardly subject to doubt (for the most part by the application of general conservation laws that are verified in the realm of microphysics); some of these properties are listed in Section 3. Of particular importance for the work of the monograph are the de Broglie relations; their experimental sources and the observational significance of their terms are examined in Subsection 3.1.

The central argument of the chapter begins in Section 4 with a study of the so-called "wave-particle dualism." Examination of "diffraction" experiments located the deficiencies of both wave and particle pictures of electrons and photons, and exhibited a number of curious consequences of the de Broglie relations:

(1) The motions of single electrons cannot be observed in detail; even the simple observation of whether an electron passes one or another of two slits cannot be effected without marked changes in the features manifested when observation is not made.

(2) The regularities in observed microphysical processes apply to ensembles rather than single systems; it is an ensemble of electrons rather than a single electron that produces a diffraction pattern.

(3) There exists a basic incompatibility between the observables, "electron position" and "electron momentum," expressed by the Heisenberg uncertainty princple (Section 5). The degree or measure of this incompatibility is given, roughly, by Planck's constant, h. For processes in which h may be considered "small," microphysical behavior may be described by classical methods.

In Section 6, alternative interpretations of the consequences of the de Broglie relations are examined. Two classically based suggestions - (a) the use of classical statistical methods, and (b) the assumption of the existence of "hidden variables" that might restore classical determinism - are rejected as unsatisfactory. A continued analysis of further consequences of the de Broglie relations leads to the realization that there is hardly a feature of the classical picture of motion that is observable in those microphysical processes for which h is not negligible. Velocities, accelerations, forces, paths, all concepts of primary importance in classical physics, simply cannot be observed in the microphysical realm when the limitations of the Heisenberg principle are significant. It is in this realm that microphysical behavior exhibits a paradoxical character. Our incapacity to observe classical properties seems to make the achievement of an understanding of microphysics very difficult, if not impossible.

This hardship stimulated a perception of fundamental significance. The deepening gloom caused by the recognition of so broad an incapacity to learn more about microphysical processes gave way to renewed optimism with the realization that what is truly unobservable requires no explanation, and, indeed, may not even exist! This thought frees us from the prison of the classical conceptual system; while we struggled ineffectually to bend the bars at the prison window, the door was opening to a new and revolutionary interpretation of the Heisenberg principle. The Heisenberg principle does not represent a limitation on our capacity to observe but rather a limitation on the validity of the classical conceptual scheme! In this interpretation the concepts derived from macrophysical experience are useful, and appear to have precise content, only because in macroprocesses the quantity, h, is wholly negligible.

Revolutions, no matter how soundly based, well intentioned, and essential to continued progress, are highly destructive in their initial phases. If a revolution is to succeed, it must look quickly to the replacement of those institutions that performed essential services in the old regime with new ones that are consistent with revolutionary goals. Unless the vitally needed reconstruction is possible within the new system, the revolution must fail. The new interpretation of the Heisenberg principle undermined a number of the concepts on which every scientific structure was believed to rest (Section 6).

After the chains that bound thought to classical ideas were broken, it was found that the classical concepts of "system property" and "state," on which classical scientific structures had been based, were in need of either extensive repair or complete replacement. Unless concepts were created to do the work normally required of the "property" and "state" concepts in classical physics, a consistent theory of microphysics, founded on the new interpretation of the Heisenberg principle, could not be formulated. Sections 7 and 8 constitute a sketch of the required creations; they indicate that meanings, closely allied to, but essentially different from classical meanings, can be assigned to the concepts of "property" and "state," and that with these new meanings there remain no apparent internal contradictions in the conceptions required for a reconstruction of theory.

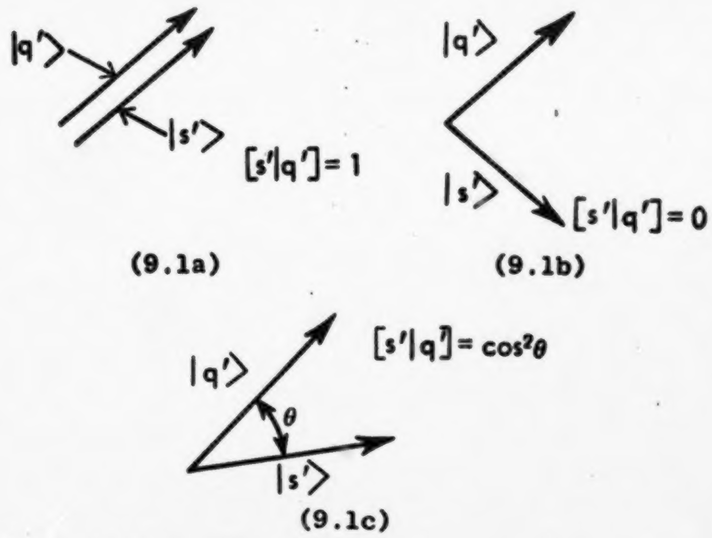
In the course of redefining the "property" and "state" concepts, two ideas of central significance for the character and content of microphysical theory were singled out for close examination: "incompatible properties" and "statistical determinism." The complex of features related to the words "indeterminacy," "unobservability," so characteristic of microphysical phenomena, may be traced to the relation of incompatibility between pairs of microphysical properties (Section 7). This new relation of incompatibility forms a bridge between the sharply differentiated relations of "compatibility" and "exclusiveness" that are familiar in classical physics. It is the incompatibility of observable properties that causes classical interpretation to fail so completely in microphysics.

The principle of statistical determinism (like the corresponding classical principle of determinism), is an inductive generalization based on the observation of the regular and lawful behavior of ensembles (for the classical case, single systems) of suitably prepared microphysical systems (Section 8). Although the behavior of individual microsystems is indeterminate, relations between the properties of ensembles may be definite and reproducible. The principle of statistical determinism makes possible the construction of consistent ideas of "state" and "state properties," and characterizes the general form of law and prediction in the realm of microphysics.

The "quantum mechanics," a logical and enormously successful theory, is founded on the concepts and attitudes described in Sections 7 and 8. This statistical theory provides a description of a broad range of the observable features of atoms, molecules, and nuclei. From the quantum mechanics the valid predictions of classical theory may be deduced.

9.2 THE MATHEMATICS OF QUANTUM MECHANICS

The enormous conceptual differences between classical and "quantum" theories are, naturally, reflected in their mathematical structures. The principle of statistical determinism implies that the quantum theory must be statistical in character; from the assumptions of the theory the determinate distributions (in state ensembles) of observables, or, equivalently, the probability of measuring any specific observable property on a system in a specified state ensemble, must be deducible. The technical device by means of which distributions and probabilities are obtained in the theory is interesting. Each distinct state ensemble, or each set of state properties (for a system) is associated with a state vector in an abstractly constructed state space. To any two state ensembles $|q'\rangle$ and $|s'\rangle$, state vectors denoted by $|q'\rangle$, $|s'\rangle$, are assigned in the state space. The probability $[q'|s']$ of finding state properties, $[q']$, if measurement is made in an ensemble $|s']$, is determined by the angle between the state vectors, $|q'\rangle$, $|s'\rangle$. If the two state vectors are colinear (Fig. 9.1a), then, according to the theory, the state properties $[q']$ and $[s']$ are compatible, (conversely, compatible state properties are associated with colinear state vectors), and the probability of observing the $[q']$ properties in the ensemble $|s']$ is unity. In other words if vectors $|q'\rangle$ and $|s'\rangle$ are parallel, the $|q']$ and $|s']$ ensembles are equivalent. If, instead, the state properties $[q']$, $[s']$ are exclusive, the associated vectors in the state space are orthogonal (Fig. 9.1b); for exclusive properties the probability of observing $[s']$ in $|q']$ is $[s'|q'] = 0$. Generally the association of state properties with vectors is such that the probability, $[s'|q']$ is given by the square of the cosine of the "angle"[67] between the vectors $|q'\rangle$ and $|s'\rangle$ (Fig. 9.1c). If the vectors $|q'\rangle$, $|s'\rangle$ are neither colinear nor orthogonal, the state-property sets $[q']$ and $[s']$ are incompatible. A vector-space scheme of the sort



(9.1a) (9.1b)

(9.1c)

Fig. 9.1 Example of the descriptions of compatible properties (a), exclusive properties (b), incompatible properties (c), by means of "vectors".

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[67] Exactly what is meant by an angle in the multidimensional, complex, state spaces of quantum mechanics has not yet been defined. The angles between state vectors are certainly not measurable (in general) with protractors. The relations between state vectors and probabilities will be investigated in Chapter 3.

sketched above could be employed by classical theory. Since, in this theory, all state sets are either compatible or exclusive, only a mutually orthogonal set of state vectors would be related to the possible states of a classical system. In quantum mechanics, pairs of state vectors that are neither colinear nor orthogonal are associated with pairs of incompatible state properties. Again, it is the presence of these incompatible properties along with the vectors related to them that distinguishes classical theory from quantum mechanics.

If q is an observable, and the exclusive properties $[q']$, $[q'']$, $[q''']$, etc., are state properties belonging to q, then the vectors associated with these exclusive properties form a system of mutually orthogonal vectors $|q'\rangle$, $|q''\rangle$, $|q'''\rangle$, etc. If the state set of properties $[s']$ is incompatible with every property of the observable, q, then the vector $|s'\rangle$ must have an orientation like that shown in Fig. 9.2. The distribution in q in the state ensemble $|s']$ is determined by the angles that $|s'\rangle$ makes with the orthogonal vectors $|q'\rangle$, $|q''\rangle$, etc. The value of the distribution at q'', for example, is the square of the cosine of the "angle" between $|q''\rangle$ and $|s''\rangle$.

The influence of incompatibility on the mathematical structure of quantum mechanics may be described in a different way. In classical theory, properties are described in terms of numbers (with associated dimensions); the position observable x can take on values from the real number continuum from $-\infty$ to $+\infty$. The numbers x', p', E' (position, momentum, energy), etc., referring to system properties, may be manipulated according to the usual algebraic rules. Such manipulations, for example, are indicated in the expression, $E' = p'^2/2m + kx'^2/2$; E' is defined by various products and a sum in the quantities x', p' and the "constants" m, k, 2. The algebra is commutative; the products xp and px are equal.

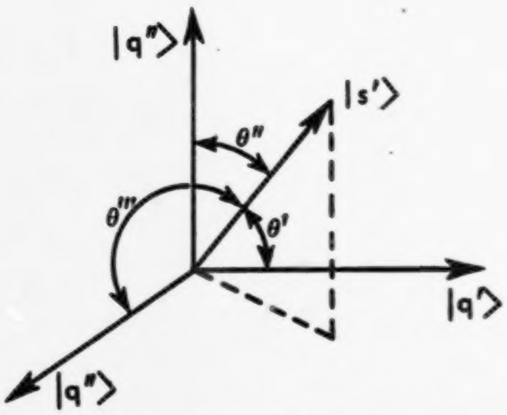

Fig. 9.2 Illustration of the computation of the distribution of the q observable in the ensemble $|s']$.

In quantum mechanics a noncommutative algebra is employed. The theory associates algebraic elements with observables as in classical theory; the position, momentum and energy, etc., (of some system) for example, are associated with elements X, P, E, etc. These elements may also be manipulated algebraically and obey the usual rules of algebra except for the commutative rule. The elements which belong to incompatible observables do not commute. From the incompatibility of position and momentum it follows that $XP \neq PX$. But the elements belonging to compatible observables do commute. Quantum mechanical and classical algebraic relations often have great formal similarity. The differences in interpretation of these relations arise in part from the noncommutativity of the elements employed by quantum mechanics. For example, the relation between E, x, p, for an oscillator in classical theory is $E = p^2/2m + \frac{1}{2}kx^2$. In quantum mechanics, the relation between the corresponding entities E, X, P, is $E = P^2/2m + \frac{1}{2}kX^2$. Classically, the expression for E may be factored:

$$\begin{aligned} E &= p^2/2m + kx^2/2 \\ &= \tfrac{1}{2}\,(p/\sqrt{m} + ix\sqrt{k})\,(p/\sqrt{m} - ix\sqrt{k}) \\ &= \tfrac{1}{2}\,(p/\sqrt{m} - ix\sqrt{k})\,(p/\sqrt{m} + ix\sqrt{k}). \end{aligned}$$

The equality of those forms depends on

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instead, it specifies merely the distribution in the position of an ensemble of electrons at a detecting screen. In accepting the successful revolutionary concepts of indeterminacy and incompatibility, physics gives up the effort to provide a "picture" of "objective reality" and limits itself to what some regard the relatively menial task of predicting the results of observations.

Consider, for example, the question, "What is an electron?" In the early sections of this chapter both the classical "wave" and "particle" models of an electron were rejected, for neither provides a satisfactory description of *all* features of diffraction experiments. A fair number of words have been devoted to describing what the electron is *not*, but the reader will search unsuccessfully through the constructive considerations of Sections 6, 7, 8 for the sentence: "Thus we see that the electron is . . . ." The observable phenomena produced by electrons are described by quantum mechanics in great detail, but to a request for a short description of an electron the theory can only reply: "The electron is an entity with a certain set of observable properties; among these there exist well-defined relations." This answer is not likely to excite the response, "Oh, so that's what an electron is."

Because of the indefiniteness of the statistical descriptions in the new theory, because of the failure of quantum mechanics to describe phenomena in terms of the *objective* characteristics of microsystems, a few of the great contributors to the theory, including Einstein, Schrödinger, and de Broglie, remained dissatisfied. As yet, however, no successful alternative to quantum mechanics has been discovered and most physicists believe that further advances in theory will not lead back to classical concepts. The meaningfulness, for example, of arbitrarily small space-time intervals has been questioned. Are distances of the order of 10^{-16} cm observable, or time intervals of 10^{-25} sec (in 10^{-25} sec a photon travels about 10^{-15} cm)? If not, are there new constants that determine limitations on the meaning of small intervals in the same sense that h measures the limitations on the concept of simultaneous position and momentum?

The future of physics cannot, of course, be predicted. However, the character of current theoretical investigations suggests that, as penetration into the microphysical domain goes on - with investigations into the interiors of "elementary" particles, and with the study of processes involving energy transformations of billions of electron volts - new revolutions will carry physics ever further from classical concepts.

9.5 CONCEPTUAL REVOLUTIONS IN PHYSICS

The history of physics includes three major conceptual revolutions. The work of the seventeenth century, which, with the publication of Newton's monumental *Principia*, gave to physics its distinctive character, revised earlier concepts of "motion," "force," "mass." Einstein's theory led to radical changes in concepts of "space," "time," and "gravitation." Quantum mechanics revised concepts of "system property" and "determinism."

These revolutions have several characteristics in common. In each case, certain observations appear to face existing theory with insuperable difficulties; within the reigning system of concepts the problems set by these observations have no solution. Instead of planning a painful and diligent search for new features of nature which might make solutions possible within old patterns of thought, the new theory lightly bypasses the problems and frames its concepts and axioms so that the offending observations are introduced into its very foundations. Instead of continuing to lose the game to nature, the physicist

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tries changing the rules of play. It takes great genius, however, to invent new rules that make for playable, profitable, and interesting games.

Before the time of Galileo and Newton, men had struggled for centuries to "explain" the motion of an arrow after it leaves a bow, or the motion of a pebble after it leaves a sling. Accepted theory, largely Aristotelian, claimed that motion without a mover is unthinkable. What, then, pushed the arrow along in its flight? Newton does not answer this question; he simply assumes, in his first law, that objects "naturally" maintain motion. The feature of motion that was significant for Newton is not constant velocity, but changes in velocity, i.e., acceleration.

Within classical conceptions of space and time, the implications of the Michelson, Morley experiment (and many others) are paradoxical. How can the speed of light be independent of the motion of an observer? Einstein doesn't answer this question; instead he takes the constancy of the velocity of light as one of the axioms of his theory. His great achievement is his proof that with this axiom a self-consistent theory that preserves the principle of relativity is possible. To build the constancy of the velocity of light into the theory required, however, highly significant revisions in classical concepts of space and time.

The application of classical theory to microphysics appeared to be blocked by the complex of observations associated with the words "wave-particle dualism." How could electrons appear to behave sometimes like waves and sometimes like particles? To classical theory, electron behavior seemed simply schizophrenic. Quantum mechanics doesn't "explain" the apparent duality. Instead, the dualism is interpreted as reflecting the inadequacy of the conceptual scheme underlying classical theory, and a new conceptual framework, within which the apparent dualism fits in a natural manner, is sought. By means of extensive changes in the concepts of "determinism" and "objective property," it is possible to construct a consistent theory into which the wave-particle characteristics of microphysical entities can be fitted. The construction has been costly; to achieve it many cherished habits of thought had to be sacrificed. "But," as the saying goes, "the recompense is ample."

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NOTE:

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